2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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9 Motion Part 3

9.1 Simple Harmonic Motion – The Differential Equation

Simple Harmonic Motion - The Differential Equation: The motion of a particle is called simple harmonic motion if its displacement from some origin satisfies $\ddot{x} = -n^2 x$, where n is a positive constant.

The Solutions of $\ddot{x} = -n^2 x$: If a particle's motion satisfies $\ddot{x} = -n^2 x$, then its displacement-time equation has the form

$$x = a\sin(nt + \alpha) \quad \text{or} \quad x = a\cos(nt + \alpha),$$

where a, n and α are constants, with $n > 0$, and $a > 0$.
In particular, the period of the motion is $T = \frac{2\pi}{n}$.

Example 9.1.1 A particle is moving so that $\ddot{x} = -4x$, in units of centimetres and seconds. Initially, it is stationary at $x = 6$.

1. *Write down the period and amplitude, and the displacement-time function.*

Solution:
Given that $\ddot{x} = -4x$, *and* $\ddot{x} = -n^2 x$, $\Rightarrow$ $n^2 = 4$, $\therefore$ $n = 2$ *and the period is* $\frac{2\pi}{2} = \pi$.

Since it starts stationary at $x = 6$, $\Rightarrow$ $a = 6$. $\therefore x = 6\cos 2t$. (*cosine starts at the maximum*).

2. *Find the position, velocity and acceleration of the particle at* $t = \frac{\pi}{3}$.

Solution: *Differentating*, $v = -12\sin 2t$, *and* $\ddot{x} = -24\cos 2t$.

$$\text{When } t = \frac{\pi}{3}, \begin{cases} \text{gives} & x = 6\cos\frac{2\pi}{3} = -3, \\ \text{and} & v = -12\sin\frac{2\pi}{3} = -6\sqrt{3}\ cm/s, \\ \text{and} & \ddot{x} = -24\cos\frac{2\pi}{3} = 12\,cm/s^2 \end{cases}$$

3. *Find* v^2 *as a function of* x, *hence find the velocity when the particle is at* $x = 3$.

Solution:
Replacing $\ddot{x}$ *by* $\frac{d}{dx}(\frac{1}{2}v^2)$, $\Rightarrow$ $\frac{d}{dx}(\frac{1}{2}v^2) = -4x$.

Then integrating, $\frac{1}{2}v^2 = -2x^2 + \frac{1}{2}C$, $v^2 = -4x^2 + C$,

when $x = 6$, $v = 0$, *so* $0 = -144 + C$, $C = 144$,

$\therefore$ $v^2 = 144 - 4x^2$, $\Rightarrow$ $v^2 = 4(36 - x^2)$.

When $x = 3$, $v^2 = 4(36 - 9) = 108$, $\Rightarrow$ $v = \sqrt{108} = \pm 6\sqrt{3}$.

Example 9.1.2 A particle P moves so that its acceleration is proportional to its displacement x from a fixed point 0 and opposite in direction. Initially the particle is at the origin, moving with velocity $12\,m/s$, and the particle is stationary when $x = 4$.

1. *Find $\ddot{x}$ and v^2 as functions of x.*

Solution: *Given that $\ddot{x} = -n^2 x$, where $n > 0$ is a constant of proportionality.*

Hence $\frac{d}{dx}(\frac{1}{2}v^2) = -n^2 x$.

Integrating, $\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + \frac{1}{2}C \;\Rightarrow\; v^2 = -n^2x^2 + C$.

When $x = 0,\; v = 12,$ *so* $144 = 0 + C,\;\Rightarrow\; C = 144$.

When $x = 4,\; v = 0,$ *so* $0 = -n^2 \times 16 + 144,$ *so* $n^2 = 9,\;\Rightarrow\; n = 3\,(n > 0)$.

Hence $\ddot{x} = -9x,$ *and* $v^2 = 9(16 - x^2)$.

2. *Find x, v and $\ddot{x}$ as functions of t.*

Solution: *Also, since the amplitude a is* 4 *and* $n = 3, x = 4\sin 3t$. (*sine starts at origin, moving uo*).

Differentiating, $v = 12\cos 3t,$ *and* $\ddot{x} = -36\sin 3t$.

3. *Find the displacement, acceleration and times when the particle is at rest.*

Solution: *Substituting* $v = 0,\;\Rightarrow\; x = \pm 4,\; \ddot{x} = \pm 36,$ *and* $t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots$

4. *Find the velocity, acceleration and times when the displacement is zero.*

Solution: *Substituting* $x = 0,\;\Rightarrow\; v = \pm 12,\; \ddot{x} = 0,$ *and* $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$

5. *Find the displacement, velocity and acceleration when $t = \frac{4\pi}{9}$.*

Solution: *Substituting* $v = 0 \;\Rightarrow\; x = \pm 4,\; \ddot{x} = \pm 36,$ *and* $t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots$

6. *Find the acceleration and velocity and times when $x = 2$.*

Solution: *Substituting* $x = 2,\;\Rightarrow\; v = \pm 6\sqrt{3},\; \ddot{x} = -18,$ *and* $t = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \ldots$

Moving the Origin of Space:

- The speed and acceleration are independent of what origin is chosen, so the velocity and acceleration functions are unchanged if the centre of motion is shifted from origin.
- A particle is moving in simple harmonic motion about $x = x_0$ if its acceleration is proportional to its displacement $x - x_0$ from $x = x_0$ but oppositely directed, that is if $\ddot{x} = -n^2(x - x_0)$, where n is a positive constant.

Example 9.1.3 A particle's motion satisfies the equation $v^2 = -x^2 + 7x - 12$. [This question involves a situation where the centre of motion is at first unknown, and must be found by expressing $\ddot{x}$ in terms of x.]

1. *Show that the motion is simple harmonic, and find the centre, period and amplitude of the motion.*

Solution:

Differentiating, $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2) = -x + 3\frac{1}{2}$,

So $\ddot{x} = -(x - 3\frac{1}{2})$, *which is in the form* $\ddot{x} = -n^2(x - x_0)$, *with* $x_0 = 3\frac{1}{2}$ *and* $n = 1$.

Hence the motion is simple harmonic, with centre $x = 3\frac{1}{2}$ *and periof* 2π

Put $v = 0$ (*to find where the particle stops at its extrems*)

When $v = 0$, $\Rightarrow$ $-x^2 + 7x - 12 = 0$, $\Rightarrow$ $x = 3$, *or* $x = 4$.

so the extremes of the motion are $x = 3$ *and* $x = 4$, *and so the ampplitude is* $\frac{1}{2}$

2. *Find where the particle is when its speed is half the maximum speed.*

Solution: *The maximum speed occurs at the centre* $x = 3\frac{1}{2}$.

Substituting into $v^2 = -(x-3)(x-4)$, $\Rightarrow$ $v^2 = \frac{1}{4}$, $\Rightarrow$ $|v| = \frac{1}{2}$.

To find where the particle is when it has half that speed, put $v = \frac{1}{4}$:

$$-x^2 + 7x - 12 = \frac{1}{16} \Rightarrow 16x^2 - 112x + 193 = 0,\ x_{1,2} = 3\frac{1}{2} \pm \frac{1}{4}\sqrt{3}.$$

Exercise 9.1.1 A particle is oscillating according to the equation $\ddot{x} = -16x$ (in units of centimetres and seconds), and its speed at the origin is $24\,cm/s$.

1. *Integrate this equation to find an equation for v^2.*

2. *What are the amplitude and the period?*

3. *Find the speed and acceleration when $x = 2$.*

Exercise 9.1.2 The acceleration of a particle moving on a horizontal line is given by $\ddot{x} = v^2 - 4v$, where v is its velocity in metres per second. The particle passes through x = 7 with a velocity of $5\,m/s$. Express v as a function of x.

Exercise 9.1.3 A particle is moving with amplitude 6 metres according to $\ddot{x} = -4x$ (the units are metres and seconds).

1. *Find the velocity-displacement equation, the period and the maximum speed.*

2. *Find the simplest form of the displacement-time equation if initially the particle is:*

   (a) *stationary at $x = 6$,*

   (b) *stationary at $x = -6$,*

   (c) *at the origin with positive velocity,*

   (d) *at the origin with negative velocity.*

Exercise 9.1.4

1. *A particle moving in simple harmonic motion on a horizontal line has amplitude 2 metres. If its speed passing through the centre O of motion is $15\,m/s$, find v^2 as a function of the displacement x to the right of O, and find the velocity and the acceleration of the particle when it is $\frac{2}{3}$ metres to the right of O.*

2. *A particle moves so that its acceleration is proportional to its displacement x from the origin O. When $4\,cm$ on the positive side of O, its velocity is $20\,cm/s$ and its acceleration is $-6\frac{2}{3}\,cm/s^2$. Find the amplitude of the motion.*

Exercise 9.1.5

1. *A particle moves in simple harmonic motion with centre O, and passes through O with speed* $10\sqrt{3}\,cm/s$*. By integrating* $\ddot{x} = -n^2x$*, calculate the speed when the particle is halfway between its mean position and a point of instantaneous rest.*

2. *A particle in a straight line obeys* $v^2 = -9x^2 + 18x + 27$*. Prove that the motion is simple harmonic, and find the centre of the motion, the period and the amplitude.*

3. *A particle's motion satisfies the equation* $v^2 = 8 - 10x - 3x^2$*. Prove that the motion is simple harmonic, and find the centre of the motion, the period and the amplitude.*

9.2 Projectile Motion – The Time Equations

Velocity and the Resolution of Velocity:

- Thevelocity of a projectile can be specified by giving its speed and angle of inclination.
- The angle of inclination is the acute angle between the path and the horizontal.
- It is positive if the object is travelling upwards, and negative if the object is travelling downwards.

Example 9.2.1 Find the horizontal and vertical components of the velocity of a projectile moving with speed 12 m/s and angle of inclination:

1. $60°$

Solution:

$$\dot{x} = 12\cos 60° = 6\, m/s.$$
$$\dot{y} = 12\sin 60° = 6\sqrt{3}\, m/s.$$

2. $-60°$

Solution:

$$\dot{x} = 12\cos 60° = 6\, m/s.$$
$$\dot{y} = -12\sin 60° = -6\sqrt{3}\, m/s.$$

Example 9.2.2 Find the speed v and angle of inclination θ (correct to the nearest degree) of a projectile for which:

1. $\dot{x} = 4\, m/s$ *and* $\dot{y} = 3\, m/s$,

Solution:

$$v^2 = 4^2 + 3^2 \;\Rightarrow\; v = 5\, m/s$$
$$\therefore\; \tan\theta = \frac{3}{4} \;\Rightarrow\; \theta \approx 37°$$

2. $\dot{x} = 5\, m/s$ *and* $\dot{y} = -2\, m/s$,

Solution:

$$v^2 = 5^2 + 2^2 \;\Rightarrow\; v = \sqrt{29}\, m/s$$
$$\therefore\; \tan\theta = -\frac{2}{5} \;\Rightarrow\; \theta \approx -22°$$

Resolution of Velocity: $\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \end{cases}$ and $\begin{cases} v^2 = \dot{x}^2 + \dot{y}^2 \\ \tan\theta = \frac{\dot{y}}{\dot{x}} \end{cases}$

The Fundamental Equations of Projectile Motion: $\ddot{x} = 0$ and $\ddot{y} = -g$

Example 9.2.3 A ball is thrown with initial velocity $40\,m/s$ and angle of inclination $30°$ from the top of a stand 25 metres above the ground.

1. *Using the stand as the origin and $g = 10m/s^2$, find the six equations of motion.*

Solution:
Initially $x = y = 0$, *and* $\begin{cases} \dot{x} = 40\cos 30° = 20\sqrt{3} \\ \dot{y} = 40\sin 30° = 20. \end{cases}$ *To begin,* $\ddot{x} = 0 \ldots (1)$

Integrating, $\dot{x} = C_1$, *when* $t = 0$, $\dot{x} = 20\sqrt{3} \Rightarrow C_1 = 20\sqrt{3} \Rightarrow \therefore \dot{x} = 20\sqrt{3} \ldots (2)$

Integrating, $x = 20\sqrt{3}t + C_2$, *when* $t = 0$, $x = 0$, $C_2 = 0$, $\Rightarrow \therefore x = 20\sqrt{3}t \ldots (3)$

To begin, $\ddot{y} = -10 \ldots (4)$

Integrating, $\dot{y} = -10t + C_3$, *when* $t = 0$, $\dot{y} = 20 \Rightarrow C_3 = 20$, $\Rightarrow \dot{y} = -10t + 20 \ldots (5)$

Integrating, $y = -5t^2 + 20t + C_4$, *when* $t = 0$, $y = 0$, $C_4 = 0$, $\therefore y = -5t^2 + 20t \ldots (6)$

2. *Find how high the ball rises, how long it takes to get there, what its speed is then, and how far it is horizontally from the stand.*

Solution: *At the top of its flight, the vertical component of the ball's velocity is zero,*

so put $\dot{y} = 0$, $\Rightarrow -10t + 20 = 0$, $\Rightarrow \therefore t = 2\,sec.$

When $t = 2$, $y = -20 + 40 = 20$ *metres. (the maximum height)*

When $t = 2$, $x = 40\sqrt{3}$ *metres. (the horizontal distance).*

Because the vertical component of velocity is zero, so $\dot{x} = 20\sqrt{3}\,m/s$.

3. *Find the flight time, the horizontal range, and the impact speed and angle.*

Solution: *Put* $y = -25$. $\Rightarrow -5t^2 + 20t = -25$, $t^2 - 4t - 5 = 0$,

so $(t-5)(t+1) = 0$, $\Rightarrow \therefore t = 5\,s$.

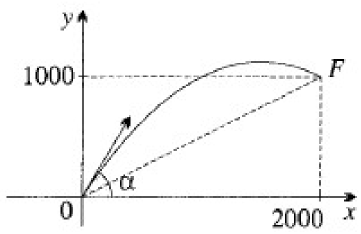
When $t = 5$ $x = 100\sqrt{3}$ *metres. also* $\dot{x} = 20\sqrt{3}$ *and* $\dot{y} = -50 + 20 = -30$,

$v^2 = 1200 + 900 \Rightarrow v = 10\sqrt{21}\,m/s$ *(the impact speed),*

$\tan\theta = -\dfrac{30}{20\sqrt{3}} = 40°54'$. *(the impact angle is about* $40°54'$*)*

Example 9.2.4 A gun at O fires shells with an initial speed of $200\,m/s$ but a variable angle of inclination α. Take $g = 10m/s^2$.

1. *Find the two possible angles at which the gun can be set so that it will hit a fortress F $2\,km$ away on top of a mountain 1000 metres high.*



Solution:

Place the origin at the gun, so that initially, $x = 0$, *and* $y = 0$.

Resolving the initial velocity, $\dot{x} = 200\cos\alpha$, *and* $\dot{y} = 200\sin\alpha$.

To begin, $\ddot{x} = 0 \ldots (1)$

Integrating, $\dot{x} = C_1$, *when* $t = 0$, $\Rightarrow$ $\dot{x} = 200\cos\alpha$,

so $C_1 = 200\cos\alpha$, $\Rightarrow$ $\dot{x} = 200\cos\alpha \ldots (2)$

Integrating, $x = 200t\cos\alpha + C_2$. *When* $t = 0$, $x = 0$, $\Rightarrow$ $C_2 = 0$,

so $x = 200t\cos\alpha \ldots (3)$

To begin, $\ddot{y} = -10 \ldots (4)$

integrating, $\dot{y} = -10t + C_3$. *when* $t = 0$, $\Rightarrow$ $\dot{y} = 200\sin\alpha$,

so $C_3 = 200\sin\alpha$, $\Rightarrow \therefore$ $\dot{y} = -10t + 200\sin\alpha \ldots (5)$

Integrating, $y = -5t^2 + 200t\sin\alpha + C_4$, *when* $t = 0$, $\Rightarrow$ $y = 0$,

so $C_4 = 0$, $\Rightarrow \therefore$ $y = -5t^2 + 200t\sin\alpha \ldots (6)$

Since the fortress is $2\,km$ *away,*, $x = 2000$,

so from (3) $200t\cos\alpha = 2000$, $\Rightarrow$ $t = \dfrac{10}{\cos\alpha}$.

Since the mountain is 1000 metres high, $y = 1000$

so from (6) $-5t^2 + 200t\sin\alpha = 1000$.

Hence $-\dfrac{500}{\cos^2\alpha} + \dfrac{2000\sin\alpha}{\cos\alpha} - 1000 = 0$, $\Rightarrow$ $\sec^2\alpha - 4\tan\alpha + 2 = 0$,

But $\sec^2\alpha = \tan^2\alpha + 1$,

so $\tan^2\alpha - 4\tan\alpha + 3 = 0$, $\Rightarrow$ $(\tan\alpha - 3)(\tan\alpha - 1) = 0$,

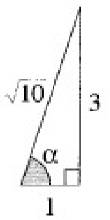
$\therefore$ $\tan\alpha = 1$ *or* 3, $\alpha = 45^\circ$ *or* $\alpha = 71^\circ 34'$.

2. *Show that the two angles are equally inclined to OF and to the vertical.*

Solution: $\angle FOX = \tan^{-1}(\frac{1}{2}) = 26°34'$, *so the* $45°$ *shot is inclined at* $18°26'$ *to OF*,

and the $71°34'$ *shot is inclined at* $18°26'$ *to the vertical.*

3. *Find the corresponding flight times and the impact speeds and angles.*



Solution: *When* $\alpha = 45°$, *from part 1* $\Rightarrow$ $t = \dfrac{10}{\cos\alpha} = 10\sqrt{2}$ *seconds*,

and when $t = 10\sqrt{2}$, *from (5)*, $\Rightarrow$ $\dot{y} = -100\sqrt{2} + 200 \times \dfrac{\sqrt{2}}{2} = 0$,

so the shell hits horizontally at $100\sqrt{2}\ m/s$.

When $\alpha = \tan^{-1} 3$, $\Rightarrow$ $\cos\alpha = \dfrac{1}{\sqrt{10}}$ *and* $\sin\alpha = \dfrac{3}{\sqrt{10}}$,

so $t = \dfrac{10}{\cos\alpha} = 10\sqrt{10}$ *seconds*,

and when $t = 10\sqrt{10}$, *from (5)* $\dot{y} = -100\sqrt{10} + 60\sqrt{10} = -40\sqrt{10}$,

and from (2), $\dot{x} = 20\sqrt{10}$,

so $v^2 = 16000 + 4000 = 20000$, $\Rightarrow$ $v = 100\sqrt{2}\, m/s$,

and $tan\theta = \dfrac{\dot{y}}{\dot{x}} = -2$,

$\theta = -\tan^{-1} 2 \approx -63°26'$,

so the shell hits at $100\sqrt{2}\, m/s$ *at about* $63°26'$ *to the horizontal.*

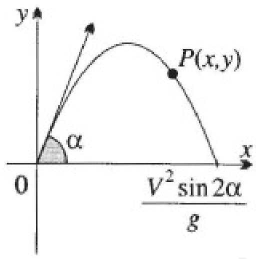
9.3 Projectile Motion – The Equation of Path

The equation of the path: The path of a projectile fired from the origin with initial speed v and angle of elevation α is

$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha. \ [Note: sec^2\alpha = 1+\tan^2\alpha.]$$

Example 9.3.1 Use the equation of path above in these questions.

1. *Show that the range on level ground is* $\frac{V^2}{g}\sin 2\alpha$*, and hence find the maximum range for a given initial speed V and variable angle* α *of elevation.*



Solution:

Put $y=0$, *then* $\frac{gx^2\sec^2\alpha}{2V^2} = x\tan\alpha$,

so $x=0$, *or* $\frac{gx}{2V^2\cos^2\alpha} = \frac{\sin\alpha}{\cos\alpha} \Rightarrow x = \frac{V^2}{g}\cos\alpha\sin\alpha$

$\therefore\ x = \frac{V^2}{g}\sin 2\alpha.$

Hence the projectile lands $\frac{V^2}{g}\sin 2\alpha$ *away from the origin.*

Since $\sin 2\alpha$ *has a maximum value of 1 when* $\alpha = 45°$,

Therefore the maximum range is $\frac{V^2}{g}$ *when* $\alpha = 45°$.

2. *Arrange the equation of path as a quadratic in* $\tan\alpha$*, and hence show that with a given initial speed V and variable angle* α *of elevation, a projectile can be fired through the point* $P(x,y)$ *if and only if* $2V^2gy \le v^4 - g^2x^2$.

Solution:

Multiplying both sides of the equation of path by $2V^2$,

$2V^2y = -gx^2(1+\tan^2\alpha) + 2V^2x\tan\alpha$

$2V^2y + gx^2 + gx^2\tan^2\alpha - 2V^2x\tan\alpha = 0$

$gx^2\tan^2\alpha - 2V^2x\tan\alpha + (2V^2y + gx^2) = 0.$

The equation has now been written as a quadratic in $\tan\alpha$.

It will have a solution for $\tan\alpha$ *provided that* $\Delta \ge 0$,

That is, $(2V^2x)^2 - 4\times gx^2\times(2V^2y+gx^2) \ge 0$

$4V^4x^2 - 8x^2V^2gy - 4g^2x^4 \ge 0 \ \Rightarrow\ 2V^2gy \le V^4 - g^2x^2.$

9.4 Practical Exam Questions

Exercise 9.4.1 A projectile is thrown downwards with a speed of $20\,m/s^2$ from the top of a cliff at an angle of 30° to the horizontal. It hits the ground after 5 seconds. Take $g = 10\,m/s^2$.

1. *Find the height of the cliff.*

2. *Find the velocity and direction when it hit the ground below the cliff.*

Exercise 9.4.2 A projectile is thrown at an angle of $30°$ at a speed of $20\,m/s$. Taking $g = 10\,m/s^2$

1. *Find its time of flight.*

2. *Find its range.*

3. *Find the greatest height.*

4. *Find its speed and direction after 1.5 seconds.*

Exercise 9.4.3 The velocity of a particle moving on a horizontal line is given by $v^2 = 9(16 - x^2)$.

1. *Show that its motion is simple harmonic motion.*

2. *Find its greatest speed.*

3. *Write down its period.*

Exercise 9.4.4 A particle is moving on a horizontal line in simple harmonic notion. The velocity of the particle is given by $v^2 = 16(8 + 2x - x^2)$**.**

1. *Find the places where the particle is stationary*

2. *Find the centre of motion.*

3. *Write down the period.*

4. *Find the amplitude.*

Exercise 9.4.5 A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \neq \frac{\pi}{2}$. You may assume that $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$, where x and y are the horizontal and vertical displacements of the projectile in metres from O and at time t seconds after firing.

1. *Show that the equation of flight of the projectile can be written as:* $y = x\tan\theta - \frac{1}{4h}x^2(1 + \tan^2\theta)$, *where* $h = \frac{V^2}{2g}$.

2. *Show that the point(X, Y), where* $X \neq 0$, *can be hit by firing at two different angles* θ_1 *and* θ_2 *provided* $X^2 < 4h(h - Y)$.

3. *Show that no point above the x axis can be hit by firing at two different angles* θ_1 *and* θ_2*, satisfying* $\theta_1 < \frac{\pi}{4}$ *and* $\theta_2 < \frac{\pi}{4}$.

Exercise 9.4.6 A projectile is thrown at 40 m/s at an angle of $\alpha = \tan^{-1}\frac{3}{4}$ **to the horizontal. Find the equation of the trajectory. Take** $g = 10\, m/s^2$**.**

Exercise 9.4.7 From a tower 60 m high on a horizontal plane, a ball is thrown upwards at an angle of $\alpha = \tan^{-1}\frac{4}{3}$ at a speed of $25\,m/s$, Take $g = -10\,m/s^2$.

1. *Find the equation of the trajectory.*

2. *Find the greatest height above the ground that the ball will reach.*

3. *Find how long before the ball hits the ground.*

4. *Find how far from the base of the tower that the ball will hit the ground.*