2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
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9 Motion Part 2

9.1 The Acceleration Due to Gravity

The Acceleration Due to Gravity:

- Since the time of Galileo, it has been known that near the surface of the Earth, a body free to fall accelerates downwards at a constant rate, and whatever its velocity (neglecting air resistance).
- This acceleration is called the acceleration due to gravity, and is conventionally given the symbol g.
- The value of this acceleration is about $9.8\,m/s^2$, or in round figures, $10\,m/s^2$.
- Choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible.

Example 9.1.1 A stone is dropped from the top of a high building. How far does it travel, and how fast is it going, after 5 seconds? (Take $g = 9.8\,m/s^2$)

Solution: *Let x be the distance travelled t seconds after the store is dropped.*

This puts the origin of space at the top of the building and the origin of time at the instant when the stone is dropped, and makes downwards positive.

Given that $a = \ddot{x} = 9.8\ m/s^2$

Integrating, $\Rightarrow\ v = 9.8t + C,$ *for some constant C.*

Since the stone was dropped, its initoal speed was zero, and

Substituting, $t = 0,\ \Rightarrow\ 0 = 0 + C\ \ \Rightarrow\ C = 0,$

$\therefore v = 9.8t.$

Integrating again, $x = \frac{1}{2} \times 9.8t^2 + D,$ *for some constant D.*

Since the intitial displacement of the stone was zero,

Substituting $x = 0,\ 0 = 0 + D\ \ \Rightarrow\ D = 0,$

$\therefore x = 4.9t^2.$

When $t = 5,\ \Rightarrow\ v = 9.8 \times 5 = 49\,m/s,$ *and* $x = 4.9 \times 5^2 = 122.5\ m.$

Example 9.1.2 A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of $40m/s$, and it falls back past the lookout onto the valley floor below. How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10m/s^2$).

Solution: *Let x be the disdance above the valley floor t seconds after the stone is thrown.*

This puts the origin of space at the valley floor

and the origin of time at the instant when the stone is thrown.

It also makes upwards positive, so that $\ddot{x} = -10$, because the acceleration is downwards.

As discussed, $\ddot{x} = -10 \ldots (1)$

Integrating, $v = -10t + C,$ *for some constant C.*

Since $v = 40,$ *when* $t = 0, \Rightarrow 40 = 0 + C \Rightarrow C = 40$

$\therefore v = -10t + 40 \ldots (2)$

Integrating again, $\Rightarrow x = -5t^2 + 40t + D,$ *for some constant D.*

Since $x = 100,$ *when* $t = 0, \Rightarrow 100 = 0 + 0 + D, \Rightarrow D = 100,$

$\therefore x = -5t^2 + 40t + 100 \ldots (3)$

The stone hits the ground when $x = 0,$ *that is:*

$$-5t^2 + 40t + 100 = 0$$

$$t^2 - 8t - 20 = 0$$

$$(t + 2)(t - 10) = 0, \Rightarrow t = -2 \text{ or } 10.$$

$\therefore$ *the ball hits the ground after 10 seconds.*

At that time, substituting $t = 10, \Rightarrow v = -100 + 40 = -60,$

Therefore the ball hits the ground at $60\,m/s.$

9.2 Using Definite Integrals to Find Changes in Displacement and Velocity

Using definite integrals to find changes in displacement and velocity:

Given velocity v as a function of time, then from $t = t_1$ to $t = t_2$,

$$\text{change in displacement} = \int_{t_1}^{t_2} v\,dt.$$

Given acceleration $\dot{x}$ as a function of time, then from $t = t_1$ to $t = t_2$,

$$\text{change in velocity} = \int_{t_1}^{t_2} \ddot{x}\,dt.$$

Example 9.2.1

1. *Given $v = 5 - e^{5-t}$, find the change in displacement in metres during the third seconds.*

Solution:

$$\begin{aligned}\textit{Change in displacement} &= \int_2^3 (5-e)^{5-t}\,dt \\ &= [5t + e^{5-t}]_2^3 \\ &= (15 + e^2) - (10 + e^3) \\ &= 5 + e^2 - e^3 \textit{ metres.}\end{aligned}$$

2. *Given $\ddot{x} = 16\sin 2t$, find the change in velocity during the first $\frac{\pi}{2}$ seconds.*

Solution:

$$\begin{aligned}\textit{Change in volocity} &= \int_0^{\frac{\pi}{2}} 16\sin 2t\,dt \\ &= -8[\cos 2t]_0^{\frac{pi}{2}} \\ &= -8(-1-1) \\ &= 16\,m/s.\end{aligned}$$

Exercise 9.2.1 A stone is dropped from a lookout 80 metres high. Take $g = 10m/s^2$, and downwards as positive, so that $\ddot{x} = 10$.

1. *Using the lookout as the origin, find the velocity and displacement as function of t.*

2. *Find the time the stone takes to fall.*

3. *Find its impact speed.*

4. *Where is it, and what is its speed, halfway through its flight time?*

5. *How long does it take to go halfway down, and what is its speed then?*

Exercise 9.2.2 A particle is moving with acceleration $\ddot{x} = 12t$. Initially it has velocity $-24m/s$, and is 20 metres on the positive side of the origin.

1. *Find the velocity and displacement functions.*

2. *When does the particle return to its initial position, and what is its speed then?*

3. *What is the minimum displacement, and when does it occur?*

4. *Find x when $t = 0, 1, 2, 3$ and 4, and sketch the displacement-time graph.*

Exercise 9.2.3 A mouse emerges from his hole and moves out and back along a line. His velocity at time t seconds is $v = 4t(t-3)(t-6) = 4t^3 - 36t^2 + 72t\,cm/s$.

1. *When does he return to his original position, and how fast is he then going?*

2. *How far does he travel during this time, and what is his average speed?*

3. *What is his maximum speed, and when does it occur?*

4. *If a video of these 6 seconds were played backwards, could this be detected?*

9.3 Simple Harmonic Motion – The Time Equations

Simple Harmonic Motion - The Time Equations:

A particle is said to be moving in simple harmonic motion with centre the origin if

$$x = a\sin(nt + \alpha) \text{ or } x = a\cos(nt + \alpha)$$

where a, n and α are constants, with a and n positive.

Simple Harmonic Motion about Other Centres:

A particle is said to be moving in simple harmonic motion with centre $x = x_0$ if

$$x = x_0 + a\sin(nt + \alpha) \text{ or } x = x_0 + a\cos(nt + \alpha)$$

The Differential Equation for Simple Harmonic Motion:

If a particle is moving in simple harmonic motion with centre the origin and period $\frac{2\pi}{n}$,

$$\text{then } \ddot{x} = -n^2 x$$

.

Example 9.3.1 A particle is moving in simple harmonic motion according to the equation $x = 2 + 4\cos(2t + \frac{\pi}{3})$**. Find the first time when the particle is at:**

1. *the centre of motion.*

Solution: *The centre is* $x = 2$, *Put* $2 + 4\cos(2t + \dfrac{\pi}{3}) = 2$,

Then $\cos(2t + \dfrac{\pi}{3}) = 0, \Rightarrow 2t + \dfrac{\pi}{3} = \dfrac{\pi}{2} \Rightarrow t = \dfrac{\pi}{12}$.

2. *the origin,*

Solution: *Put* $2 + 4\cos(2t + \dfrac{\pi}{3}) = 0$, *then* $\cos(2t + \dfrac{\pi}{3}) = -\dfrac{1}{2}$

so $\Rightarrow 2t + \dfrac{\pi}{3} = \dfrac{2\pi}{3} \Rightarrow t = \dfrac{\pi}{6}$.

3. *the maximum and minimum displacement.*

Solution:

The centre is $x = 2$ *and the amplitude is* 4, *so the motion therefore lies in the interval* $-2 \le x \le 6$.

Put $2 + 4\cos(2t + \dfrac{\pi}{3}) = 6$, *then* $\cos(2t + \dfrac{\pi}{3}) = 1$ *so* $\Rightarrow 2t + \dfrac{\pi}{3} = 2\pi \Rightarrow t = \dfrac{5\pi}{6}$.

Put $2 + 4\cos(2t + \dfrac{\pi}{3}) = -2$, *then* $\cos(2t + \dfrac{\pi}{3}) = -1$ *so* $\Rightarrow 2t + \dfrac{\pi}{3} = \pi \Rightarrow t = \dfrac{\pi}{3}$.

Example 9.3.2 Suppose that a particle is moving according to $x = 2\sin(3t + \frac{3\pi}{4})$**.**

1. *Write down the amplitude, centre, period and initial phase of the motion.*

Solution:
The amplitude is 2*, the centre is* $x = 0$*, the period is* $\dfrac{2\pi}{3}$*, and the initial phase is* $\dfrac{3\pi}{4}$.

2. *Find the times and positions when the velocity is first: (i) zero, (ii) maximum.*

Solution:
Differentiating, $v = 6\cos\left(3t + \dfrac{3\pi}{4}\right)$*, so the maximum velocity is 6.*

(i) When $v = 0,\ \cos\left(3t + \dfrac{3\pi}{4}\right) = 0,\ \Rightarrow\ 3t + \dfrac{3\pi}{4} = \dfrac{3\pi}{2},\ \Rightarrow\ t = \dfrac{\pi}{4}.$

When $t = \dfrac{\pi}{4},\ x = 2\sin\dfrac{3\pi}{2} = 2.$

(ii) When $v = 6,\ 6\cos\left(3t + \dfrac{3\pi}{4}\right) = 6,\ \Rightarrow\ 3t + \dfrac{3\pi}{4} = 2\pi\ \Rightarrow\ t = \dfrac{5\pi}{12}.$

When $t = \dfrac{5\pi}{12},\ x = 2\sin 2\pi = 0.$

3. *Find the times and positions when* $\ddot{x}$ *is first: (i) zero, (ii) maximum.*

Solution:
Differentiating, $\ddot{x} = -18\sin\left(3t + \dfrac{3\pi}{4}\right)$*, so the maximum acceleration is* 18.

(i) When $\ddot{x} = 0,\ \sin\left(3t + \dfrac{3\pi}{4}\right) = 0,\ \Rightarrow\ 3t + \dfrac{3\pi}{4} = \pi\ \Rightarrow\ t = \dfrac{\pi}{12}.$

When $t = \dfrac{\pi}{12},\ \Rightarrow\ x = 2\sin\pi = 0.$

(ii) When $\ddot{x} = 18,\ -18\sin\left(3t + \dfrac{3\pi}{4}\right) = 18,\ \Rightarrow\ 3t + \dfrac{3\pi}{4} = \dfrac{3\pi}{2}\ \Rightarrow\ t = \dfrac{\pi}{4}.$

When $t = \dfrac{\pi}{4},\ \Rightarrow\ x = 2\sin\dfrac{3\pi}{2} = -2.$

4. *Express the acceleration* $\ddot{x}$ *as a multiple of the displacement* x.

Solution:
Since $x = 2\sin\left(3t + \dfrac{3\pi}{4}\right)$ *and* $\ddot{x} = -18\sin(3t + \dfrac{3\pi}{4}),$

it follows that $\ddot{x} = -9x.$

Since $n = 3,$ *this agrees with the general result* $\ddot{x} = -n^2x.$

Choosing the Origin of Time and the Function: Try to make the initial phase zero:

- Use $x = a\cos nt$ if the particle starts at the positive extreme of its motion, and use $x = -a\cos nt$ if the particle starts at the negative extreme.
- Use x = a sin nt if the particle starts at the middle of its motion with positive velocity, and use $x = -a\sin nt$ if the particle starts at the middle of its motion with negative velocity.
- If the particle starts anywhere else, try to change the origin of time. Otherwise, use $x = a\cos(nt + \alpha)$ or $x = a\sin(nt + \alpha)$, and then substitute the boundary conditions to find α and a.

Example 9.3.3 A weight hanging from the roof on an elastic string is moving in simple harmonic motion. It takes 4 seconds to move from the bottom of its motion, $15\,cm$ above the floor , to the top of its motion, $55\,cm$ above the floor.

1. *Find where it is 3 seconds after rising through the centre of motion.*

Solution: *Choose the origin at the centre of motion, 35 cm above the ground.*
Choose the origin of time at the instant it passes through the centre, moving upwards.
Then the amplitude is 20 cm and the period is $2 \times 4 = 8$ seconds
So $n = \frac{2\pi}{8} = \frac{\pi}{4}$, *and* $x = 20\sin\frac{\pi}{4}t$.
When $t = 3,\ x = 20\sin\frac{3\pi}{4} = 10\sqrt{2}$,
therefore the weight is $35 + 10\sqrt{2}\,cm$ *above the groud.*

2. *Find its speed at the centre of motion.*

Solution: *Differentiating,* $v = 5\pi\cos\frac{\pi}{4}t$, *The weight is at the centre when* $t = 0$,
and then speed $= 5\pi\cos 0 = 5\pi\ cm/s$.

3. *Find the maximum acceleration.*

Solution: *differentiating again,* $\ddot{x} = -\frac{5\pi^2}{4}\sin\frac{\pi}{4}t$,
so the maximum accelration is $\frac{5\pi^2}{4}\ cm/s^2$.

The Graphs of $x = a\cos(nt+\alpha)$ and $x = a\sin(nt+\alpha)$:
When the initial phase α is nonzero, the graphs can be sketched by shifting the graphs of

$$x = a\cos nt \text{ and } x = a\sin nt.$$

The key step here is to take out the factor of n and write

$$x = a\cos n(t+\tfrac{\alpha}{n}) \text{ and } x = a\sin n(t+\tfrac{\alpha}{n})$$

These graphs are $x = a\cos nt$ and $x = a\sin nt$ shifted left be $\frac{\alpha}{n}$.

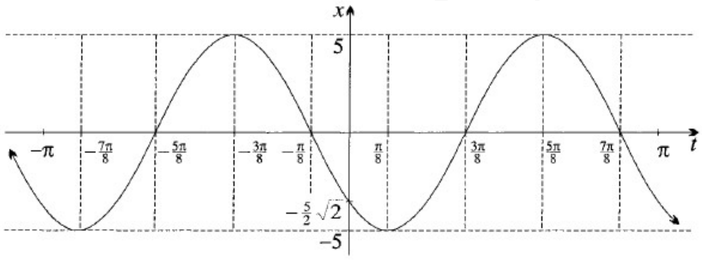
Example 9.3.4

1. *Sketch* $x = 5\cos(2t+\frac{3\pi}{4})$, *for* $-\pi \le t \le \pi$.

Solution: $x = 5\cos(2t+\dfrac{3\pi}{4}) \Rightarrow x = 5\cos 2(t+\dfrac{3\pi}{8})$

This is a cosine wave with amplitude 5 and period π, *shifted left by* $\dfrac{3\pi}{8}$ *units.*

Also when $t = 0$, $x = 5\cos\dfrac{3\pi}{4} = -\dfrac{5\sqrt{2}}{2}$.



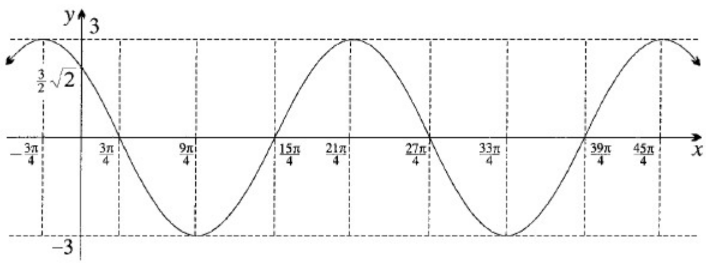
2. *Sketch* $y = -3\sin(\frac{1}{3}x-\frac{\pi}{4})$.

Solution: $y = -3\sin(\dfrac{1}{3}x-\dfrac{\pi}{4}) \Rightarrow y = -3\sin\dfrac{1}{3}(x-\dfrac{3\pi}{4})$,

This is $y = 3\sin\dfrac{1}{3}(x-\dfrac{3\pi}{4})$ *reflected in the x-axis.*

This is sine wave with amplitude 3, *and period* 6π, *shifted righ by* $\dfrac{3\pi}{4}$.

Also when $x = 0$, $y = -3\sin(-\dfrac{\pi}{4}) = \dfrac{3\sqrt{2}}{2}$.



9.4 Motion Using Functions of Displacement

Velocity as a Function of Displacement: If the velocity is given as a function of displacement, take the reciprocal to give $\frac{dt}{dx}$ as a function of x, and then integrate with respect to x.

Example 9.4.1 Suppose that a particle is initially at the origin, and moves according to $v = e^{-x}\,m/s$. Find x, v and $\ddot{x}$ in terms of t, and find how long it takes for the particle to travel 1 metre. Briefly describe the subsequent motion.

Solution:

Given that: $\frac{dx}{dt} = e^{-x}$,

$\frac{dt}{dx} = e^{x}, \Rightarrow t = e^{x} + C$

When $t = 0,\ x = 0$

$0 = 1 + C$,

so $C = -1$, *and* $t = e^{x} - 1$,

and it takes $e - 1$ *seconds to go 1 metre.*

Solving for x, $e^{x} = t + 1,\ x = \ln(t+1)$.

Differentiating, $v = \frac{1}{t+1}$

and $\ddot{x} = -\frac{1}{(t+1)^2}$.

The particle moves to infinity.

Its velocity remains positive,

decreases with limit zero.

Example 9.4.2 A particle moves so that its velocity is proportional to its displacement from the origin O. Initially it is 1 cm to the right of the origin, moving to the left with a speed of $0.5\,cm/s$. Find the displacement and velocity as functions of time, and briefly describe its motion.

Solution:

$v = kx$, *for some constant* k

then $x = 1,\ v = -\frac{1}{2}, \Rightarrow$ *so* $-\frac{1}{2} = k \times 1$,

so $k = -\frac{1}{2}$, *and* $v = -\frac{1}{2}x$.

Takibng reciprocals, $\frac{dt}{dx} = -\frac{2}{x}$

and intgrating, $t = -2\ln x + C$, *for some constant C.*

When $x = 1,\ t = 0, \Rightarrow \therefore\ 0 = 0 + c$,

so $C = 0$, *and* $t = -2\ln x$.

Solveing for x, $\Rightarrow x = e^{1-\frac{1}{2}t}$

and differentiating, $v = -\frac{1}{2}e^{-\frac{1}{2}t}$

Thus the particle continues to move to left, its speed decreasing with limit zero, and the origin begin its limiting positions.

Acceleration as a Function of Displacement: If the acceleration is given as a function of displacement, use the form $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ for acceleration, and then integrate with respect to x.

Example 9.4.3 Suppose that a ball attached to the ceiling by a long spring will hang at rest at the point $x = 0$. The ball is lifted 2 metres above $x = 0$ and dropped, and subsequently moves according to the equation $\ddot{x} = -4x$. Find its speed as a function of x, and show that it comes to rest 2 metres below $x = 0$. Find its maximum speed and the place where this occurs.

Solution: *Given that:* $\ddot{x} = -4x, \Rightarrow \dfrac{d}{dx}(\dfrac{1}{2}v^2) = -4x.$

Integrating with respect to x, $\Rightarrow \dfrac{1}{2}v^2 = -2x^2 + \dfrac{1}{2}C,$ *for some constant C,*

$v^2 = -4x^2 + C.$

When $x = 2,\ v = 0,$ *so* $0 = -16 + C, \Rightarrow C = 16, \Rightarrow \therefore\ v^2 = 16 - 4x^2.$

Hence $v = 0$ *when* $x = -2,$ *as required.*

When $x = 0, \Rightarrow$ *the maximum speed is* $4m/s$ *when* $x = 0,$

Exercise 9.4.1 A particle is moving with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.

1. *Find v^2 as a function of displacement.*

2. *Assuming that v is never positive, find the displacement as a function of time, and briefly describe the motion, mentioning what happens as $t \to \infty$.*

Exercise 9.4.2 In each case, the acceleration $\ddot{x}$ is given as a function of x. By replacing $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$ and integrating, express v^2 in terms of x, given that $v = 0$ when $x = 0$.

1. $\ddot{x} = 6x^2$

2. $\ddot{x} = \sin 6x$.

3. $\ddot{x} = \frac{1}{4+x^2}$.

Exercise 9.4.3 The acceleration of a particle P is given by $\ddot{x} = -2x$ (in units of centimetres and seconds), and the particle starts from rest at $x = 2$.

1. *Find the speed of P when it first reaches $x = 1$, and explain whether it must then be moving backwards or forwards.*

2. *In what interval is the motion confined, and what is the maximum speed?*

Exercise 9.4.4 A plane lands on a runway at $100\,m/s$. It then brakes with a constant deceleration until it stops 2 km down the runway.

1. *Explain why the equation of motion is $\ddot{x} = -k$, for some positive constant k. By integrating with respect to x, find k, and find v^2 as a function of x.*

2. *Find the velocity after $1\,km$,*

3. *Find where it is when the velocity is $50\,m/s$.*

4. *Explain why, during the braking, $v = \sqrt{10000 - 5x}$ rather than $v = -\sqrt{10000 - 5x}$.*

5. *Integrate to find the displacement-time function, and find how long it takes to stop.*

Exercise 9.4.5

1. *A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is* v. *At time* $t = 0, x = 0$ *and* $v = V$. *Subsequently the particle is slowing down at a rate equal to* kv^3, *where* k *is a positive constant. Show that* $t = \frac{x}{V} + \frac{1}{2}kx^2$.

2. *A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is* v. *At time* $t = 0,\ x = 0$ *and* $v = 1$. *Subsequently the particle experiences a retardation of magnitude* e^v . *Find the time* t_1 *for the particle to slow to half its initial and the further time* t_2 *for the particle to come to rest. Deduce that* $\frac{t_2}{t_1} = e^{\frac{1}{2}}$.