2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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9 Motion Part 1

9.1 Average Velocity and Speed

- **Motion in One Dimension:** Motion in one dimension is specified by given the displacement x on the number line as a function of time t after time zero.
- **Average Velocity:** average velocity = $\frac{\text{Change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$.
  On the displacement-time graph, average velocity = gradient of the chord.
- **Distance Travelled:** Distance travelled is always positive or zero, and takes into account any journey and return.
- **Average Speed:** average speed = $\frac{\text{distance travelled}}{\text{time taken}}$.

Example 9.1.1 Suppose that a ball is thrown vertically upwards from ground level, and lands 4 seconds later in the same place. Its motion can be described approximately by the following equation and table of values:

$$x = 5t(4 - t)$$

t	0	1	2	3	4
x	0	15	20	15	0

1. *Given that $x = 5t(4 - t)$, at what time is the ball $8\frac{3}{4}$ metres above the ground?*

Solution:

$$\text{Put } x = 8\frac{3}{4} \;\Rightarrow 8\frac{3}{4} = 5t(4-t), \;\Rightarrow\; 4t^2 - 16 + 7 = 0,$$
$$\Rightarrow\; (2t-1)(27-7) = 0, \;\Rightarrow\; t = \frac{1}{2} \text{ or } 3\frac{1}{2} \text{ seconds}.$$

2. *Find the average velocities of the ball during the first and during the third second.*

Solution:

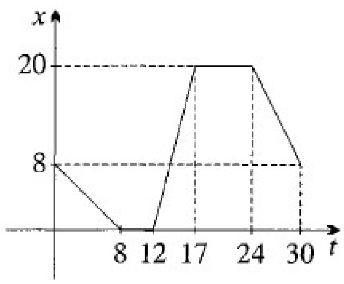
$$\text{Velocity during 1st second} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{15 - 0}{1 - 0} = 15\, m/s.$$
$$\text{Velocity during 3rd second} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{15 - 20}{3 - 2} = -5\, m/s.$$

3. *Find the average velocity and average speed of the ball during the last three seconds.*

Solution:

$$\text{Average Velocity} = \frac{\text{Change in displacement}}{\text{change in time}} = \frac{0 - 15}{4 - 1} = -5\, m/s.$$
$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{5 + 5 + 15}{4 - 1} = \frac{25}{3} = 8\frac{1}{3}\, m/s.$$

Exercise 9.1.1 Emma is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x metres from the front gate after t seconds.



1. *What is her velocity: (i) during the first 8 seconds, (ii) while she is driving forwards, (iii) while she is reversing the second time?*

2. *Find the total distance travelled, and the average speed, over the 30 seconds.*

3. *Find the change in displacement, and the average velocity, over the 30 seconds.*

4. *Find her average speed if she had not paused at the gate and at the garage.*

9.2 Displacement and Distance

Example 9.2.1 A particle moves so that its distance from the origin in given by $x = t^2 - 2t + 2$

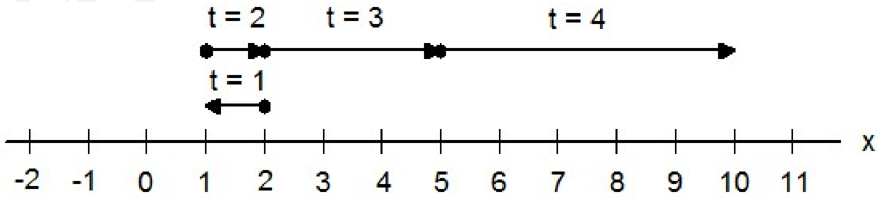
1. *What is the displacement after 4 seconds?*

Solution: *the displacement or position is obtained by simply substituting 4 into the formula* $x = t^2 - 2t + 2$.
$\therefore\ x = 4^2 - 2 \times 4 + 2 = 10\,units$
$\therefore$ *the displacement is 10 units from the origin.*

2. *What is the distance travelled during the first 4 seconds?*

Solution: *the distance travelled during the first 4 seconds can only be obtained by looking at the path of the particle on the number line.*

t	0	1	2	3	4	5
x	2	1	2	5	10	17



$\therefore$ *the total distance travelled is 10 units*

Exercise 9.2.1 If the displacement of a particle from a point is given by $x = t^2 - 4t + 3$, find:

1. *The initial position,*

2. *When the particle passed through the origin,*

3. *The displacement after 4 seconds,*

4. *The distance travelled during the first 4 seconds,*

5. *The instantaneous velocity after 4 seconds,*

6. *the average speed during the first 4 seconds.*

Exercise 9.2.2 A particle is moving on a horizontal number line according to the equation $x = 4\sin\frac{\pi}{6}t$, in units of metres and seconds.

1. *Sketch the displacement-time graph.*

2. *How many times does the particle return to the origin by the end of the first minute?*

3. *Find at what times it visits $x = 4$ during the first minute.*

4. *Find how far it travels during the first 12 seconds, and its average speed in that time.*

5. *Find the values of x when $t = 0, t = 1$ and $t = 3$. Hence show that its average speed during the first second is twice its average speed during the next 2 seconds.*

9.3 Instantaneous Velocity as Derivatives

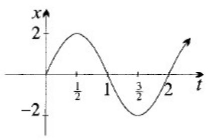
Instantaneous Velocity and Speed: The instantaneous velocity v of the particle is the derivative of the displacement with respect to time: $v = \frac{dx}{dt}$, or $\dot{x}$.

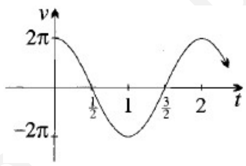
Stationary Point: To find when a particle is stationary, put $v = 0$ and solve for t.

Example 9.3.1 A particle is moving according to the equation $x = 2\sin \pi t$

1. *Find the equation for its velocity, and graph both equations.*

Solution: *Given that ,* $x = 2\sin \pi t$, *differentiating,* $v = 2\pi \cos \pi t$, *and the graph are drawn opposite.*





2. *Find when the particle is at the origin, and its speed then.*

Solution: *when th eparticle is at the origin,* $x = 0$ $2\sin \pi t = 0$, $t = 0, 1, 2, 3\dots$

When $t = 0, 2$, $v = 2\pi$, *and when* $t = 1, 3\dots v = -2\pi$.

Hence the particle is at the origin when $t = 0, 1, 2, 3\dots$ *and the speed is always* 2π.

3. *Find when and where the particle is stationary.*

Solution: *When the particle is stationary,* $v = 0$, $\Rightarrow$ $2\pi \cos \pi t = 0$, $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \dots$

When $t = \frac{1}{2}, 2\frac{1}{2}$, $x = 2$, *and when* $t = 1\frac{1}{2}, 3\frac{1}{2}$, $x = -2$.

Hence the particle is stationary when $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \dots$*and is 2 unit right of left of the origin.*

4. *Briefly describe the motion.*

Solution: *The particle oscillates for ever between* $x = -2$ *and* $x = 2$ *with period 2, beginning at the origin, and moving first to* $x = 2$.

9.4 Acceleration as Derivatives

Acceleration as a second derivative: Acceleration is the first derivative of velocity with respect to time, and the second derivative of displacement:
Instantaneous acceleration: $a = \frac{dv}{dt}$ or $a = \frac{d^2x}{dt^2} = \dot{v} = \ddot{x}$.

Note:

- Acceleration is sometimes denoted by $\ddot{x}$, where the two dots show double differentiation with respect to t.
- the units of acceleration are "metres per second per second" or m/s^2
- A negative acceleration means the particle is slowing down or decelerating

Example 9.4.1 A point moves so that its distance x metres from a fixed point 0 at time t seconds is given by $x = \frac{t^3}{3} - \frac{6t^2}{2} + 8t + 7$

1. *Find the initial displacement.*

Solution: *Initial displacement when* $t = 0 \Rightarrow \therefore x = 7\,m$

2. *Find the equations for velocity and acceleration.*

Solution: *velocity* $= v = \frac{dx}{dt} = 3\frac{t^2}{3} - 2 \times \frac{6t}{2} + 8 = t^2 - 6t + 8$
acceleration $= a = \frac{dv}{dt} = 2t - 6$

3. *At what times is the velocity zero?*

Solution: *Solving for* $v = 0$ *in the velocity equation,* $0 = t^2 - 6t + 8 \Rightarrow 0 = (t-2)(t-4)$
$\therefore\ t = 2$ *second or* $t = 4$ *seconds its velocity is zero.*

4. *What is the acceleration at these times?*

Solution: *When* $t = 2$, *acceleration* $a = 2 \times (2) - 6 = -2m/s^2$
When $t = 4$, *acceleration* $a = 2 \times (4) - 6 = 2m/s^2$

Exercise 9.4.1 A particle moves in a straight line such that its position (x metres) form a fixed point O on the line at time t seconds is given by $x = t^3 - 3t + 12$ find:

1. *Its initial position, velocity and acceleration,*

2. *the time when its velocity is zero,*

3. *What is its position and acceleration at that time?*

Exercise 9.4.2 A particle moves in a straight line so the its displacement $x(t)$ from a fixed point in the line at any time $(t \geq 0)$ is given by $x(t) = 4 + 4t - 5\sqrt{t^2 + 4}$. find the displacement when the particle comes to rest.

Exercise 9.4.3 A particle in a straight line so that its position x metres from the origin at any time t second is given $x = t^2 - 10t + 16$. find:

1. *its initial position,*

2. *its initial velocity,*

3. *When it first passes through origin and with what velocity,*

4. *when it passes through origin for the second time and with what velocity,*

5. *when and where its velocity is zero.*

Exercise 9.4.4 A ball is projected vertically upwards with an initial velocity of $30\,m/s$. It rises with a deceleration of $10\,m/s^2$. find:

1. *its velocity at any time t,*

2. *its height, h metre above the point of projection at any time t,*

3. *the greatest height reached,*

4. *the time taken to return to the point of projection.*

9.5 Integrating with Respect to Time

- The velocity can be found by differentiate the displacement function with respect to t. $v = \frac{dx}{dt}$.
- The Acceleration can be found by differentiate the displacement function twice with respect to t $a = \frac{d^2x}{dt}$.
- Integrate acceleration to obtain velocity plus a constant term $\int a\,dt = v + c$.
- Integrate velocity to obtain displacement plus a constant term $\int v\,dt = x + c$.
- An initial condition is needed to evaluate each constant of integration.

Example 9.5.1 A particle moves along the x axis with an acceleration given by $a = 12t - 2$. If initially, the particle is at $x = 3$ with a velocity of 2 metres per seconds, find the position of the particle after 5 seconds.

Solution:

Given that $a = 12t - 2$ *when* $t = 0, \Rightarrow x = 3\,m$ *and* $v = 2\,m/s$

Step 1 Integrate acceleration to find velocity formula:

$$\therefore\ v = \int (12t - 2)\,dt = \frac{12t^2}{2} - 2t + c_1$$

$$\therefore\ v = 6t^2 - 2t + c_1, \text{ when } t = 0, \Rightarrow\ v = 2$$

Substitute $t = 0$ *and* $v = 2$ *to find* c_1

$$\therefore\ 2 = 0 - 0 + c_1,$$

$$\therefore\ v = 6t^2 - 2t + 2$$

Step 2 Integrate velocity to find displacement formula:

$$\therefore\ displacement\ = \int (6x^2 - 2t + 2)\,dt$$

$$x = \frac{6t^3}{3} - \frac{2t^2}{2} + 2t + c_2$$

$$= 2t^3 - t^2 + 2t + c_2$$

Substitute $t = 0$ *and* $x = 3$ *to find* c_2.

$$\therefore\ 3 = 0 - 0 + 0 + c_2\ \Rightarrow\ c_2 = 3$$

$$\therefore\ x = 2t^3 - t^2 + 2t + 3$$

Step 3 substitute $t = 5$ *to find position after 5 seconds.*

$$\therefore\ x = 2 \times 5^3 - 5^2 + 2 \times 5 + 3 = 238.$$

$\therefore$ *the particle is* 238 *metres from origin.*

Exercise 9.5.1 The velocity of a particle initially at the origin is $v = \sin \frac{1}{4}t$.

1. *Find the displacement function.*

2. *Find the acceleration function.*

3. *Find the values of displacement, velocity and acceleration when $t = 4\pi$.*

4. *Briefly describe the motion, and sketch the displacement-time graph.*

9.6 Miscellaneous Exercise

Exercise 9.6.1 The acceleration of a particle is given by $\ddot{x} = e^{-2t}$, and the particle is initially stationary at the origin.

1. *Find the velocity function.*

2. *Find the displacement function.*

3. *Find the displacement when $t = 10$.*

4. *Briefly describe the velocity of the particle as time goes on.*

Exercise 9.6.2 A particle is moving so that its velocity t seconds after $t = 0$ seconds is $v = 4t - 4$ meter per second. Initially it is at $x = 2$ metres.

1. *Integrate, substituting the initial condition to find the displacement function.*

2. *Find when the particle is at the origin and its velocity at that time.*

3. *Explain why the particle is never on the negative side of the origin.*

4. *Differentiate to find the acceleration, and show that it is a constant.*

Exercise 9.6.3 A particle's acceleration function is $a = 6t$. Initially it is at the origin, moving with velocity -3 m/s.

1. *Integrate, substituting the initial condition, to find the velocity function.*

2. *Integrate again to find the displacement function.*

3. *Find when the particle is stationary and find its displacement at that time.*

4. *Find when the particle returns to the origin and the acceleration at that time.*

Exercise 9.6.4 A stone is dropped from a lookout 125 metres high. Take $g = 10\,m/s^2$ and downwards as positive, so that the acceleration function is $a = 10\,m/s^2$.

1. *Using the lookout as the origin, find the velocity and displacement as function of t.*

2. *Show that the stone takes 5 seconds to fall, and find its impact speed.*

3. *Where is it, and what is its speed halfway through its flight time?*

4. *Show that it takes $\sqrt{10}$ seconds to go $\frac{2}{5}$ down, and find its speed then.*

Exercise 9.6.5 A particle is moving with acceleration $\ddot{x} = 12t$. Initially, it has velocity $-24\, m/s$ is 20 metres on the positive side of the origin.

1. *Find the velocity function $\dot{x}$ and the displacement function x.*

2. *When does the particle return its initial position and what it its speed then?*

3. *Find x when $t = 0, 1, 2, 3\, and\, 4$, and sketch the displacement-time graph.*

4. *What is the minimum displacement, and when does it occur?*

9.7 Practical Exam Questions

Exercise 9.7.1 A particle is moving along the x axis. its position at time t is given by $x = t + \sin t$.

1. *At what times during the period $0 < t < 3\pi$ is the particle stationary?*

2. *At what time during the period $0 < t < 3\pi$ is the acceleration equal to 0?*

3. *Carefully sketch the graph of $x = t + \sin t$ for $0 < t < 3\pi$*

Exercise 9.7.2 A particle moves along a straight line so that its displacement, x metres, from a fixed point O is given by $x = 1 + 3\cos 2t$, where t is the measured in seconds.

1. *What is the initial displacement of the particle?*

2. *Sketch the graph of x as a function of the t for $0 \leq t \leq \pi$.*

3. *Hence, find when and where the particle first comes to rest after $t = 0$.*

4. *Find when the particle reaches its maximum speed. What is the speed at that time?*

Exercise 9.7.3 A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is given by $x = t^3 - 3t^2 + 6$

1. *Differentiate to find the velocity as a function of* t*, and differentiate again to find a.*

2. *Find the initial velocity and position.*

3. *Find the time when its velocity is zero,*

4. *What is its position and acceleration at that time?*