2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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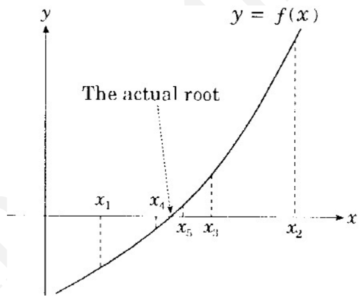
8 Numerical Estimation of the roots of an Equation

8.1 Introduction

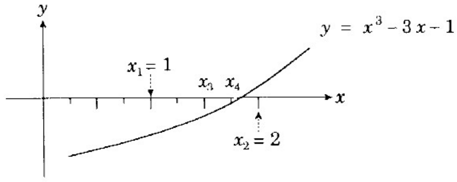
- Quadratic equations can be solved by factorisation or by using the quadratic formula.
- Polynomial equations of degree higher than two can be solved by **Factor Theorem**.
- Approximation method is used to solve trigonometric functions, logarithmic and exponential functions.
- Approximation methods of solving such equations involve staring with an estimate of the roots and then improving on the initial estimate by an **Iterative Process**, that is, one in which a sequence of steps is repeated leading to an improved approximation of the root.

8.2 Halving the Interval



- Suppose $f(x)$ is continuous in the interval $x = x_1$ to $x = x_2$.
- If $f(x_1)$ and $f(x_2)$ have opposite signs, then the root lies between $x = x_1$ and $x = x_2$ ($x_1 < 0$ and $x_2 > 0$).
- the midpoint of the interval $[x_1, x_2]$ is $x_3 = \frac{x_1+x_2}{2}$ and $f(x_3)$ is evaluated.
- $\begin{cases} \text{If } f(x_3) = 0 & \text{then } x = x_3 \text{ is the requires root.} \\ \text{If } f(x_3) < 0 & \text{then the root lies between } x_3 \text{ and } x_2 \\ \text{If } f(x_3) > 0 & \text{then the root lies between } x_3 \text{ and } x_1.\ x_3 \text{ is closer to the root.} \end{cases}$
- Repeated the process with x_1 and x_3, leading to x_4 which is still closer to the root.
- Repeated application will give the root to any desired degree of accuracy.

Example 8.2.1 Find a root of the equation $x^3 - 3x - 1 = 0$ in the interval $1 < x < 2$ using the halving method three times.



Solution: $f(x_1) = f(1) = 1 - 3 - 1 = -3$ *(negative).*

$f(x_2) = f(2) = 8 - 6 - 1 = 1$ *(positive).*

Since $f(1)$ *and* $f(2)$ *have opposite signs, there must be a root between* $x_1 = 1$ *and* $x_2 = 2$

Let $x_3 = \frac{x_1 + x_2}{2} = \frac{1+2}{2} = 1.5$

$f(1.5) = 1.5^3 - 3(1.5) - 1 = -2.125,$ *(negative)*

Therefore the root lies between $x_3 = 1.5$ *and* $x_2 = 2$.

Let $x_4 = \frac{x_3 + x_2}{2} = \frac{1.5+2}{2} = 1.75$

$f(1.75) = 1.75^3 - 3(1.75) - 1 = -0.89,$ *(negative)*

Therefore the root lies between $x_4 = 1.75$ *and* $x_2 = 2$.

Let $x_5 = \frac{x_4 + x_2}{2} = \frac{1.75+2}{2} = 1.875$

$f(1.875) = 1.875^3 - 3(1.875) - 1 = -0.03,$ *(negative)*

Therefore the root lies between $x_5 = 1.875$ *and* $x_2 = 2$.

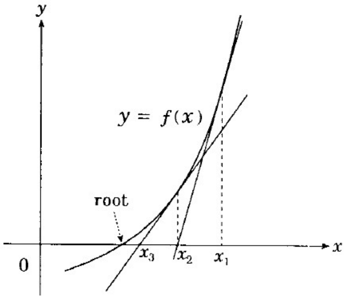
Therefore the approximate value of the roots is : $x = \frac{2 + 1.875}{2} = 1.94.$

Exercise 8.2.1 Find the root of the equation $x^3 - 3x^2 + x - 4 = 0$ in the interval $3 < x < 3.5$ using the halving method three times.

8.3 Newton's Method

Definition:

- Suppose x_1 is close to the root of the equation $f(x) = 0$.



The equation of the tangent at x_1:

$$y - y_1 = m(x - x_1)$$
$$y - f(x_1) = f'(x_1)(x - x_1)$$

- Let this tangent cross the x axis at $x = x_2$ and $y = 0$.
  Therefore $0 - f(x_1) = f'(x_1)(x_2 - x_1), \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

- If x_1 was close to the root, then it can be seen from the graph above that x_2 is even closer. Similarly, the process can be repeated to find $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

- It can be seen that x_3 will be even closer to the root that x_2.

Example 8.3.1 If $x^3 - 3x - 20 = 0$ has a root near $x = 3$, use Newton's method twice to find a better approximation.

Solution: *We take $x_1 = 3$ to be the first approximation :*

$$f(x) = x^3 - 3x - 20 \Rightarrow \therefore f(x_1) = f(3) = (3)^3 - 3(3) - 20 = -2$$
$$f'(x) = 3x^2 - 3, \Rightarrow \therefore f'(3) = 3(3^2) - 3 = 24$$
$$\text{Using } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow \therefore x_2 = 3 - \frac{-2}{24} = 3.083.$$

We take $x_2 = 3.083$ to be the second approximation:

$$f(x_2) = f(3.083) = (3.083)^3 - 3(3.083) - 20 = 0.054$$
$$f'(x_2) = f'(3.083) = 3(3.083^2) - 3 = 25.515$$
$$\text{Using } x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow \therefore x_3 = 3.083 - \frac{0.054}{25.515} = 3.081.$$

Exercise 8.3.1

1. *Use Newton's method twice to calculate* $\sqrt{18}$ *to two decimal places.*

2. *Use Newton's method twice to solve the equation* $x^2 = \sin x$ *which has a root near* $x = 1$.

3. *Given that the equation* $\ln x = 3 - x$ *has a root near* $x = 2$, *use Newton's method once to obtain a better approximation of the root.*

Exercise 8.3.2

1. *Find an approximate root for the equation* $2x^3 - x + 3 = 0$ *in the interval* $-2 < x < -1$*, to 2 decimal places.*

2. *Use Newton's method once to find an approximation to 2 decimal places of the cube root of 100.*

3. *Using Newton's method find a better approximation to the root of the equation* $\frac{x}{2} = \sin x$*. Use* $x = 2$ *as a first approximate value.*

Exercise 8.3.3

1. *Given that the equation* $\cos x = \ln x$ *has a root between 1 and 1.5, find a better approximations to 2 decimal places.*

2. *Solve the equation* $2e^{-x} - x = 0$ *correct to 2 decimal places, give that* $x = 0.8$ *is a rough root.*

3. *Show that the equation* $e^x = \cos^{-1} x$ *has a root between 0.2 and 0.3. Use Newton's method to obtain an approximation of the root correct to 3 significant figures.*

Exercise 8.3.4

1. *If x_1 is a first approximation ti the root of the equation $x^2 = c$, show that a better one, x_2 is $\frac{x_1^2+c}{2x_1}$.*

2. *Use the result in (1) to find in fraction form an approximation for $\sqrt{200}$ by taking $c = 200$ and $x_1 = 14$.*