2/3 Unit Math Homework for Year 12

Student Name: __________	**Grade**: __________
Date: __________	**Score**: __________

Table of contents

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6 Approximations of Definite Integrals

6.1 The Trapezoidal Rule

- In many cases we could not able to find the primitive of $f(x)$.
- Using the formula for the area of a trapezium as $\frac{1}{2}$ the sum of the parallel sides times the distance between them.
- The error can be reduced by subdividing the interval into n sub-interval of equal width $h = \frac{b-a}{n}$.

The Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\,[f(a) + f(b)]$$

$$\int_a^b y\,dx \approx \frac{h}{2}\,[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n]$$

where $h = \frac{b-a}{n}$ and n is the number of sub-intervals (trapezia).

Example 6.1.1 Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using the trapezoidal rule by dividing the interval [0, 2] into:

1. *2 sub-intervals.*

x	0	1	2
f(x)	2	1.732	0

Solution: *Since $h = \frac{2-0}{2} = 1$, the width h of each strip is 1 unit.*

$$\begin{aligned}\therefore \int_0^2 \sqrt{4-x^2}\,dx &\approx \frac{1}{n}[f(0) + 2f(1) + f(2)]\\ &= \frac{1}{2}[2 + 3.464 + 0]\\ &= 2.732\end{aligned}$$

2. *4 sub-intervals.*

x	0	0.5	1	1.5	2
f(x)	2	1.936	1.732	1.323	0

Solution: *Since $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$, the width h of each strip is 0.5 units.*

$$\begin{aligned}\therefore \int_0^2 \sqrt{4-x^2}\,dx &\approx \frac{h}{n}[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]\\ &= \frac{1}{4}[2 + 3.872 + 3.464 + 2.646 + 0]\\ &= 2.996\end{aligned}$$

6.2 Simpson's Rule

- The trapezoidal rule approximates the curve with a s series of straight lines on the interval $a \le x \le b$.
- Simpson's rule replace the given function by a quadratic function which intersects the given curve at $x = a$, $x = b$ and $x = \frac{a+b}{2}$.
- The area under the parabola through these three points is: $\frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$.

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_a^b y\,dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

where $h = \frac{b-a}{n}$ and n is the number of sub-intervals.

Example 6.2.1 Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using Simpson's rule with:

1. *3 function values*

x	*0*	*1*	*2*
f(x)	*2*	*1.732*	*0*

Solution: *Taking* $a = 0$, $b = 2$ *and* $\frac{a+b}{2} = \frac{0+2}{2} = 1$, *and* $h = \frac{b-a}{2} = \frac{2-0}{2} = 1$, *we get*

$$\int_0^2 \sqrt{4-x^2}\,dx \approx \frac{2-0}{6}[f(0) + 4f(1) + f(2)] = \frac{1}{3}[2 + 6.928 + 0] = 2.976$$

2. *5 function values*

x	*0*	*0.5*	*1*	*1.5*	*2*
f(x)	*2*	*1.936*	*1.732*	*1.323*	*0*

Solution: *5 function value means 4 sub-intervals,* $h = \frac{b-a}{4} = \frac{2-0}{4} = 0.5$

$$\therefore \int_0^2 \sqrt{4-x^2}\,dx \approx \frac{h}{3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] = \frac{0.5}{3}[2 + 4\times 1.936 + 2\times 1.732 + 4\times 1.323 + 0] = 3.083$$

Note: the differs from an exact value of π *by less than 2%.*

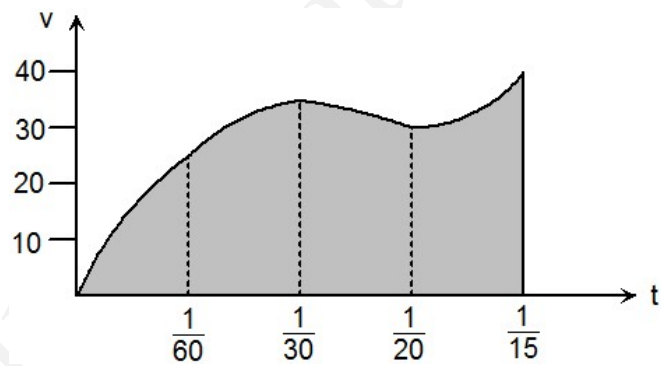
6.3 Practical Exam Questions

Exercise 6.3.1 The speed of a train recoded at intervals of one minute. The times,in minutes and the corresponding speed v, in kilometer per hour are listed in the table below.

time (min)	*0*	*1*	*2*	*3*	*4*
v (km/h)	*0*	*25*	*34*	*30*	*40*

1. *Explain why the distance x,in kilometer, travelled by the train in these four minutes is given by :*

$$x = \int_0^{\frac{1}{15}} v\,dt$$

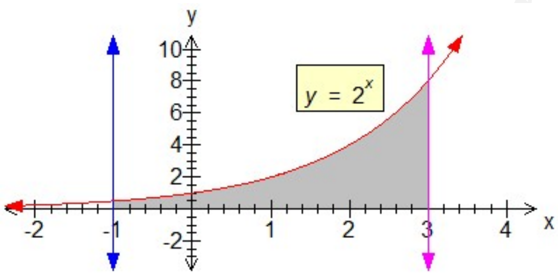


2. *Estimate x by Simpson's Rule with five function values.*

Exercise 6.3.2 Consider the function $y = 2^x$.

1. *Complete the following table of value for* $y = 2^x$.

2. *Using the Simpson's rule with these five function values, find an estimate for the area shaded in the diagram below.*



Exercise 6.3.3 Use Simpson's rule with three function values to find an Approximation for $\int_2^6 \frac{x}{log_e x}\, dx$**. Give your answer correct to one decimal place.**

6.4 Miscellaneous exercise

Exercise 6.4.1 Evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$**.**

1. *Complete the table of value for* $y=\sqrt{1-x^2}$.

2. *By using Simpson's rule with 5 function values, estimate the integral* $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.

Exercise 6.4.2 Let $f(x)=\sqrt{25-x^2}$**.**

1. *Complete the table of value for* $f(x)=\sqrt{25-x^2}$

2. *Use these six values of the function and trapezoidal rule to find the approximate value of*

$$\int_0^5 \sqrt{25-x^2}\,dx.$$

Exercise 6.4.3 The table shows the values of a function $f(x)$ for five values of x. Use Simpson's rule with these five values to estimate $\int_1^3 f(x)\,dx$

x	1	1.5	2	2.5	3
$f(x)$	5	1	-2	3	7

Exercise 6.4.4 The following table lists the values of a function for three values of x. Use these function values to estimate $\int_1^3 f(x)\,dx$ by:

x	1.0	2.0	3.0
$f(x)$	1.7	9.0	4.3

1. *The trapezoidal rule.*

2. *Simpson's rule.*

Exercise 6.4.5 The first three terms of an arithmetic series are $-1, +4 + 9 + ...$

1. *Find the 60th term.*

2. *Hence, or otherwise, find the sum of the first 60 terms of the series.*

Exercise 6.4.6 Consider the geometric series $1 + (\sqrt{5} - 2) + (\sqrt{5} - 2)^2 + ...$

1. *Explain why the geometric series has a limiting sum.*

2. *Find the exact value of the limiting sum. Write you answer with a rational denominator.*

Exercise 6.4.7 Consider the geometric series $2 + \frac{2}{\sqrt{2}+1} + \frac{2}{(\sqrt{2}+1)^2}$

1. *Find the limiting sum of the geometric series.*

2. *Explain why the geometric series* $2 + \frac{2}{\sqrt{2}-1} + \frac{2}{(\sqrt{2}-1)^2}$ *does not have a limiting sum.*

6.5 Properties of Definite Integrals

Properties of Definite Integrals:

(1.) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(2.) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(3.) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ If $f(x)$ is an even function.

(4.) $\int_{-a}^a f(x)\,dx = 0$ If $f(x)$ is an odd function.

(5.) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(6.) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where c is a constant.

(7.) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$, since x and y are dummy variables of a definite interval.