2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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5 Integration Part 3 Area and Volume

5.1 Rules of Differentiation and Integration

$f(x)$	$f'(x)$	$f(x)$	$\int f(x).dx$
kx	k	k	$kx + c$
x^n	nx^{n-1}	x^n	$\frac{x^{n+1}}{n+1} + c$
$ax^n + bx + c$	$anx^{n-1} + b$	ax^n	$\frac{ax^{n+1}}{n+1} + c$
$f\left(h(x)\right)$	$f'\left(h(x)\right).h'(x)$	$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$
$g(x).h(x)$	$g'(x).h(x) + g(x).h'(x)$	$f'(x).(f(x))^n$	$\frac{1}{n+1}(f(x))^{n+1} + c$
$y = uv$	$y' = uv' + u'v$		
$\frac{g(x)}{h(x)}$	$\frac{g'(x).h(x)-g(x).h'(x)}{[h(x)]^2}$		
$y = \frac{u}{v}$	$y' = \frac{u'v-uv'}{v^2}$		
e^x	e^x	e^x	$= e^x + c$
e^{kx}	ke^{kx}	e^{kx}	$\frac{1}{k}e^{kx} + c$
$\ln\ x$	$\frac{1}{x}$	$\frac{1}{x}$	$\ln\ x + c$
$\ln\ kx$	$\frac{k}{x}$	$\frac{k}{x}$	$\frac{1}{k}\ln\ x + c$
$\ln\ f(x)$	$\frac{f'(x)}{f(x)}$	$\frac{f'(x)}{f(x)}$	$\ln\ f(x) + c$
$\sin\ x$	$\cos\ x$	$\sin\ x$	$-\cos\ x + c$
$\sin\ kx$	$k\cos\ kx$	$\sin\ kx$	$-\frac{1}{k}\cos\ kx + c$
$\cos\ x$	$-\sin\ x$	$\cos\ x$	$\sin\ x + c$
		$\tan x$	$-\ln(\cos x) + c$
		$\cot x$	$\ln(\sin x) + c$
		$\sec x$	$\ln(\sec x + \tan x) + c$
$\cos\ kx$	$-k\ \sin\ kx$	$\cos\ kx$	$\frac{1}{k}\sin\ kx + c$
$\tan\ x$	$\sec^2\ x$	$\sec^2 x$	$\tan\ x + c$
$\tan\ kx$	$k\ \sec^2\ kx$	$\sec^2\ kx$	$\frac{1}{k}\tan\ kx + c$
$\sin^{-1}\ \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\ \frac{x}{a}c + c$
$\cos^{-1}\ \frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$		
$\tan^{-1}\ \frac{x}{a}$	$\frac{a}{a^2+x^2}$	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\ \frac{x}{a} + c$

5.2 Indefinite Integra Review

Standard Forms:

$$\int (ax^n + x^n + b)\,dx = \frac{ax^{n+1}}{n+1} + \frac{x^{n+1}}{n+1} + bx + c, \text{ where a, b, c are constants, } n \neq -1$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \text{ where a, b, c are constants, } n \neq -1$$

Example 5.2.1 Find the Primitive Function of $f(x) = \frac{3x-1}{x^3}$.

Solution:

$$\begin{aligned}\int \frac{3x-1}{x^3}\,dx &= \int 3x^{-2} - x^{-3}\,dx \\ &= \frac{3x^{(-2+1)}}{-2+1} - \frac{x^{(-3+1)}}{-3+1} + c \\ &= \frac{3x^{-1}}{-1} - \frac{x^{-2}}{-2} + c \\ &= \frac{-3}{x} + \frac{1}{2x^2} + c.\end{aligned}$$

Exercise 5.2.1 Find the following indefinite integrals:

1. $\int (5 - \frac{1}{x^4})\,dx$

2. $\int (3x - \frac{3}{x})^2\,dx$

Exercise 5.2.2 Find the following indefinite integrals:

1. $\int (1 - \frac{x}{3})^5 \, dx$

2. $\int \frac{x^2+3x}{\sqrt{x}} \, dx$

3. $\int (1 - x^2)^2 \, dx$

4. $\int \frac{4}{5(1-4x)^2} \, dx$

5. $\int \frac{x^2-1}{\sqrt[3]{x}} \, dx$

5.3 Definite Integra Review

The fundamental Theorem (Integral Form):

$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

Example 5.3.1 $\int_1^4 (x-1)\,dx$

Solution:

$$\begin{aligned}\int_1^4 (x-1)\,dx &= \left[\frac{1}{2}x^2 - x\right]_1^4 \\ &= [\frac{1}{2}(4)^2 - (4)] - [\frac{1}{2}(1)^2 - (1)] \\ &= (8-4) - (\frac{1}{2} - 1) \\ &= 4\frac{1}{2}\end{aligned}$$

Exercise 5.3.1

1. $\int_{-2}^2 (x^2+4)\,dx$

2. $\int_1^2 \frac{1}{x^2}\,dx$

Exercise 5.3.2 Evaluate the following:

1. $\int_0^3 (x+1)^3\,dx$

2. $\int_2^3 (2x-5)^3\,dx$

3. $\int_0^1 \sqrt{4-3x}\,dx$

4. $\int_1^6 \frac{1}{\sqrt{3+x}}\,dx$

5.4 Area Bounded by the x-axis and y-axis

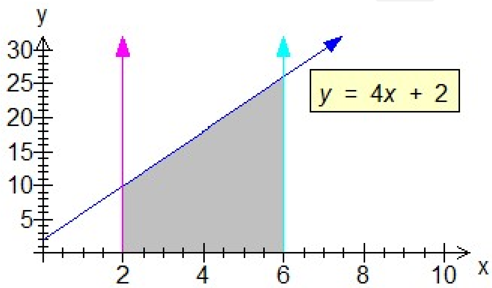
Definition: Area bounded by the x-axis and y-axis

$$A_x = \left| \int_a^b y\, dx \right| \qquad A_y = \left| \int_a^b x\, dy \right|$$

Example 5.4.1 Calculate the area of the region bounded by the graph of the straight line $y = 4x+2$, the X-axis and the ordinates x = 2 and x = 6.

Solution: *The required area measure is* $\int_2^6 (4x+2)\,dx = \left[\frac{4x^2}{2} + 2x\right]_2^6$

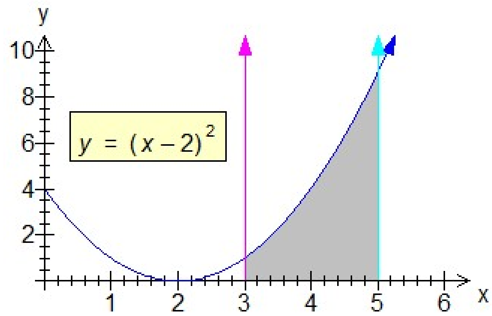
$$= [2x^2+2x]_2^6$$
$$= (2(6)^2+2(6)) - (2(2)^2+2(2))$$
$$= (72+12) - (8+4) = 72\,units^2$$



Example 5.4.2 Find, by integration the area of the region enclosed between the parabola $y = (x-2)^2$, the x-axis and the ordinates x = 3 and x= 5.

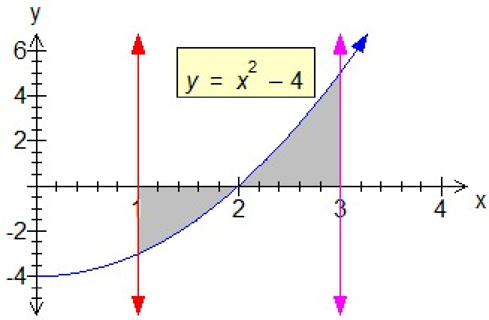
Solution: $\int_3^5 (x-2)^2\,dx = \left[\frac{(x-2)^3}{3}\right]_3^5$

$$= \frac{(5-2)^3}{3} - \frac{(3-2)^3}{3} = 9 - \frac{1}{27} = 8\frac{26}{27}\,units^2$$

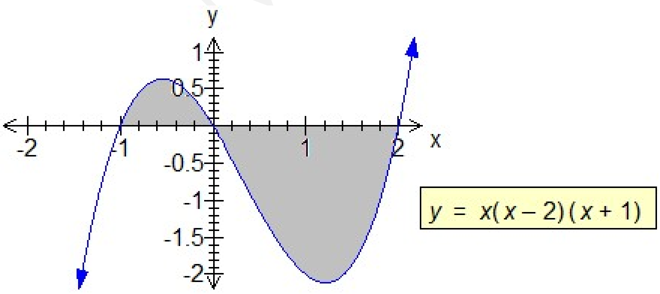


Exercise 5.4.1

1. *Find the area of the region bounded by the curve* $y = x^2 - 4$*, the x-axis and between the ordinates at x= 1 and x = 3.*



2. *Find the area, bounded by the graph of* $f(x) = x(x-2)(x+1)$ *and the x-axis.*



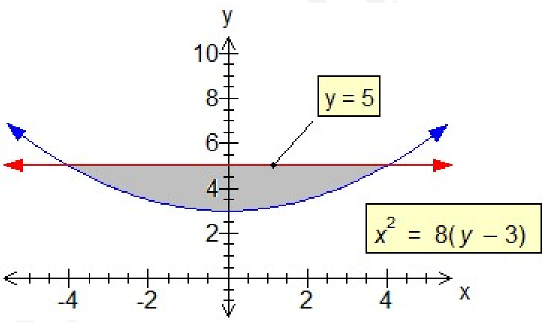
5.5 Practical Exam Questions

Exercise 5.5.1 Consider the parabola $x^2 = 8(y - 3)$

1. *Write down the coordinates of the vertex.*

2. *Find the coordinates of the focus.*

3. *Sketch the parabola.*



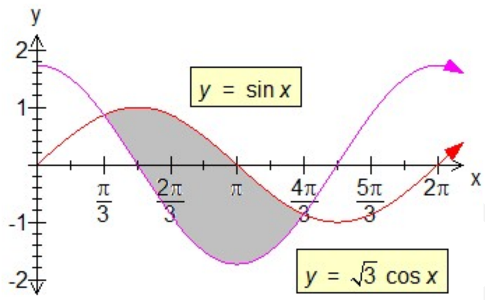
4. *Calculate the area bounded by the parabola and the line* $y = 5$

5.6 Area Between Two Curves

Definition: Area between the curves $y = f(x)$ and $y - fg(x)$

$$A = \left| \int_a^b [f(x) - g(x)]\, dx \right|$$

Example 5.6.1 The diagram shows the graphs of $y = sin\, x$ and $y = \sqrt{3}\, cos\, x$. The two point of intersection of the right of the y-axis are labelled A and B.



1. *Solve the equation $sin\, x = \sqrt{3}\, cos\, x$ to find the x-coordinates of A and B.*

Solution: $sin\, x = \sqrt{3}\, cos\, x$ *divide both sides by* $sin\, x$:
We have $\frac{sin\, x}{cos\, x} = \sqrt{3}$, $\quad tan\, x = \sqrt{3}$
$\therefore x = \frac{\pi}{3},\ \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
Therefore x-coordinates of A and B are $\frac{\pi}{3}$ and $\frac{4\pi}{3}$ respectively.

2. *Find the area of the shaded region in the diagram.*

Solution: *Area* $= \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} (sin\, x - \sqrt{3}\, cos\, x)\, dx$

$$= \left[-cos\, x - \sqrt{3}\, sin\, x\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$$
$$= -\left[cos\, x + \sqrt{3}\, sin\, x\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$$
$$= -\left[\left(cos\, \tfrac{4\pi}{3} + \sqrt{3} sin\, \tfrac{4\pi}{3}\right) - \left(cos\, \tfrac{\pi}{3} + \sqrt{3} sin\, \tfrac{\pi}{3}\right)\right]$$
$$= -\left[\left(-cos\, \tfrac{\pi}{3} - \sqrt{3} sin\, \tfrac{\pi}{3}\right) - \left(cos\, \tfrac{\pi}{3} + \sqrt{3} sin\, \tfrac{\pi}{3}\right)\right]$$
$$= -\left(-cos\, \tfrac{\pi}{3} - \sqrt{3}\, sin\, \tfrac{\pi}{3} - cos\, \tfrac{\pi}{3} - \sqrt{3}\, sin\, \tfrac{\pi}{3}\right)$$
$$= -\left(-2\, cos\, \tfrac{\pi}{3} - 2\sqrt{3}\, sin\, \tfrac{\pi}{3}\right)$$
$$= 2(\tfrac{1}{2}) + 2\sqrt{3}(\tfrac{\sqrt{3}}{2})$$
$$= 1 + 3$$
$$= 4\ units^2$$

5.7 Volume of Revolution

Definition: If A_x is rotated about the x-axis through one revolution

$$V_x = \pi \int_a^b y^2\,dx \qquad where\ y = f(x)$$

Example 5.7.1 Find the volume of a sphere of radius r.

Solution: *The Volume of the sphere may be considered as the volume generated by rotating the semicircle defined by $y = \sqrt{r^2 - x^2}$, where $-r \le x \le r$ about the x-axis.*

Hence $V = \pi \int_{-r}^{r} y^2\,dx$ where $y = \sqrt{r^2 - x^2}$

$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$ *since $y^2 = r^2 - x^2$*

$= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$

$= \pi \left[(r^3 - \frac{r^3}{3}) - (-r^3 + \frac{r^3}{3})\right]$

$= \frac{4}{3}\pi r^3\ units^3$

Definition: If A_y is rotated about the y-axis through one revolution

$$V_y = \pi \int_a^b x^2\,dy$$

Example 5.7.2 The part of the parabola $y = x^2$ between $x = 1$ and $x = 3$ is rotated about the y-axis. Find the volume generals.

Solution: *A typical layer cut perpendicular to the y-axis would be approximately cylindrical in shape with radius x and width δy and the volume δV given by $\delta V = \pi x^2 \delta y$.*

Since $y = x^2$, when $x = 1$, $y = x^2 = 1^2 = 1$; when $x = 3$, $y = 3^2 = 9$;

Therefore the total volume V would be given by $V = \pi \int_1^9 x^2 dy$

$V = \pi \int_1^9 x^2\,dy$

$= \pi \int_1^9 y\,dy$ *since $x^2 = y$*

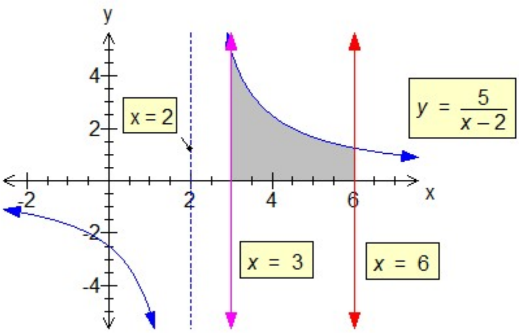
$= \pi \left[\frac{y^2}{2}\right]_1^9$

$= \pi(\frac{9^2}{2} - \frac{1}{2})$

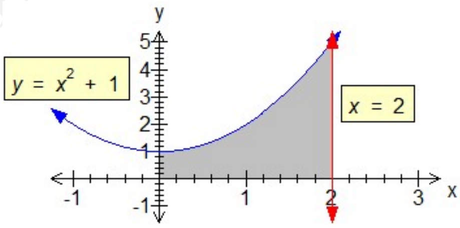
$= 40\pi\ units^3$

5.8 Practical Exam Questions

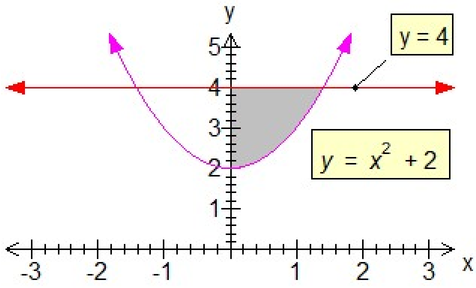
Exercise 5.8.1 The graph of $y = \frac{5}{x-2}$ is shown below. The shaded region in the diagram is bounded by the curve $y = \frac{5}{x-2}$, the x-axis and the lines $x = 3$ and $x = 6$. Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis.



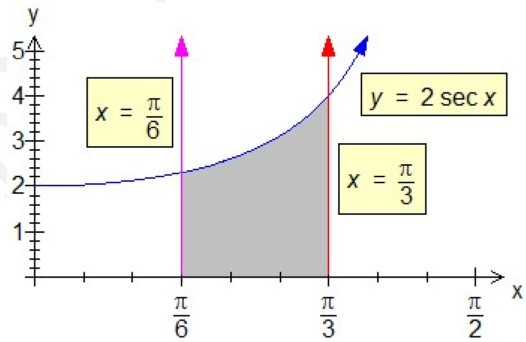
Exercise 5.8.2 The shaded region i the diagram is bounded by the curve $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 2$. Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis.



Exercise 5.8.3 In the diagram, the shaded region is bounded by the parabola $y = x^2 + 2$, the y-axis and the line y = 4. Find the volume of the solid formed when the shaded region is rotated about the y-axis.

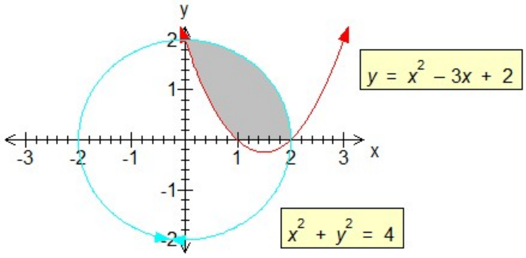


Exercise 5.8.4 In the diagram, the shaded region is bounded by the curve $y = 2\sec x$, the x-axis, the line $x = \frac{\pi}{6}$ and the line $x = \frac{\pi}{3}$. The shaded region is rotated about the x-axis. Calculate the exactly volume of the solid of revolution formed.

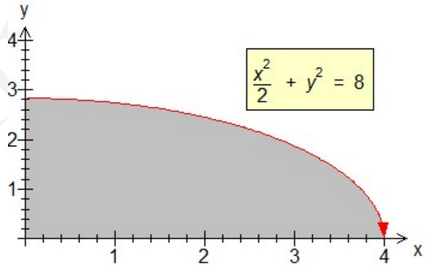


5.9 Miscellaneous exercise

Exercise 5.9.1 The shaded region in the diagram is bounded by the circle of radius 2 centred at the origin, the parabola $y = x^2 - 3x + 2$, and the x-axis. By considering the difference of two area, find the area of the shaded region.



Exercise 5.9.2 The part of the curve $\frac{x^2}{2} + y^2 = 8$ that lies in the first quadrant is rotated about the x axis. Find the volume of the solid of revolution.



Exercise 5.9.3

1. *Differentiate* $x\sin^{-1}x + \sqrt{1-x^2}$.

2. *Find the area between* $y = \sin x$, *the y-axis and the lines* $y = \frac{1}{\sqrt{2}}$ *and* $y = \frac{1}{2}$.

Exercise 5.9.4

1. *Differentiate* $x\tan^{-1}x$ *and hence evaluate* $\int_0^1 \tan^{-1}x\,dx$.

2. *Use the substitution* $x = \frac{3}{2}\sin\theta$ *to find* $\int \frac{1}{\sqrt{9-4x^2}}\,dx$.