2/3 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

Table of contents

This edition was printed on June 23, 2014 with worked solutions.
Camera ready copy was prepared with the LaTeX2e typesetting system.

2 Indefinite Integral

2.1 Rules of Differentiation and Integration

$f(x)$	$f'(x)$	$f(x)$	$\int f(x).dx$
kx	k	k	$kx + c$
x^n	nx^{n-1}	x^n	$\frac{x^{n+1}}{n+1} + c$
$ax^n + bx + c$	$anx^{n-1} + b$	ax^n	$\frac{ax^{n+1}}{n+1} + c$
$f\left(h(x)\right)$	$f'\left(h(x)\right).h'(x)$	$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$
$g(x).h(x)$	$g'(x).h(x) + g(x).h'(x)$	$f'(x).(f(x))^n$	$\frac{1}{n+1}(f(x))^{n+1} + c$
$y = uv$	$y' = uv' + u'v$		
$\frac{g(x)}{h(x)}$	$\frac{g'(x).h(x) - g(x).h'(x)}{[h(x)]^2}$		
$y = \frac{u}{v}$	$y' = \frac{u'v - uv'}{v^2}$		
e^x	e^x	e^x	$= e^x + c$
e^{kx}	ke^{kx}	e^{kx}	$\frac{1}{k}e^{kx} + c$
$\ln\ x$	$\frac{1}{x}$	$\frac{1}{x}$	$\ln\ x + c$
$\ln\ kx$	$\frac{k}{x}$	$\frac{k}{x}$	$\frac{1}{k}\ln\ x + c$
$\ln\ f(x)$	$\frac{f'(x)}{f(x)}$	$\frac{f'(x)}{f(x)}$	$\ln\ f(x) + c$
$\sin\ x$	$\cos\ x$	$\sin\ x$	$-\cos\ x + c$
$\sin\ kx$	$k\cos\ kx$	$\sin\ kx$	$-\frac{1}{k}\cos\ kx + c$
$\cos\ x$	$-\sin\ x$	$\cos\ x$	$\sin\ x + c$
		$\tan x$	$-\ln(\cos x) + c$
		$\cot x$	$\ln(\sin x) + c$
		$\sec x$	$\ln(\sec x + \tan x) + c$
$\cos\ kx$	$-k\ \sin\ kx$	$\cos\ kx$	$\frac{1}{k}\sin\ kx + c$
$\tan\ x$	$\sec^2\ x$	$\sec^2 x$	$\tan\ x + c$
$\tan\ kx$	$k\ \sec^2\ kx$	$\sec^2\ kx$	$\frac{1}{k}\tan\ kx + c$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}c + c$
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$		
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$

2.2 Indefinite Integral (Primitive Function)

Definition:

$$\int (ax^n + x^n + b)\, dx = \frac{ax^{n+1}}{n+1} + \frac{x^{n+1}}{n+1} + bx + c,\ \ where a, b, c are constants, n \neq -1$$

Definition:

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, where a, b, c are constants, n \neq -1$$

Example 2.2.1 Find the following indefinite integrals:

1. $\int 3x^2 + 2x - 4\, dx$

Solution: $\int 3x^2 + 2x - 4\, dx = \frac{3x^3}{3} + \frac{2x^2}{2} - 4x + c = x^3 + x^2 - 4x + c$

2. $\int (3x+2)^3 dx$

Solution: $\int (3x+2)^3\, dx = \frac{(3x+2)^4}{3(3+1)} + c = \frac{(3x+2)^4}{12} + c$

3. $\int \sqrt{x}\, dx$

Solution: $\int \sqrt{x}\, dx = \int x^{(\frac{1}{2})}\, dx = \frac{x^{(\frac{1}{2}+1)}}{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c$

4. $\int (5 - \frac{1}{x^4})\, dx$

Solution: $\int (5 - \frac{1}{x^4})\, dx = \int (5 - x^{-4})\, dx = 5x - \frac{x^{(-4+1)}}{(-4+1)} + c = 5x + \frac{x^{-3}}{3} + c = 5x + \frac{1}{3x^3} + c$

Exercise 2.2.1 Find the following indefinite integrals:

1. $\int \frac{4x-1}{x^3}\,dx$

2. $\int \frac{4}{\sqrt[3]{x}}\,dx$

3. $\int \frac{1}{\sqrt{x+1}}\,dx$

4. $\int (3x-2)(x+3)\,dx$

5. $\int (2x-1)^3\,dx$

2.3 Integration Involving Logarithms

Definition:

$$\int \frac{1}{x}\,dx = log_e x + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = log_e\,[f(x)] + c$$

Definition:

$$\int e^x\,dx = e^x + c; \qquad \int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + c$$

Example 2.3.1 Find the following indefinite integrals:

1. $\int \frac{1}{3x+1}\,dx$

Solution: *Here* $f(x) = 3x + 1$ *so* $f'(x) = 3$, *We will have to adjust the numerator:*

$$\int \frac{1}{3x+1}\,dx = \int \frac{1}{3} \times \frac{3}{3x+1}\,dx = \frac{1}{3}\int \frac{3}{3x+1}\,dx = \frac{1}{3}\,log_e\,(3x+1) + c.$$

2. $\int \frac{2x}{x^2-1}\,dx$

Solution: *Here* $f(x) = x^2 - 1$ *so* $f'(x) = 2x$, *No adjustments are necessary:*

$$\int \frac{2x}{x^2-1}\,dx = log_e\,(x^2-1) + c.$$

3. $\int 2 + e^{3x}\,dx$

Solution:

$$\int 2 + e^{3x}\,dx = 2x + \frac{1}{3}\,e^{3x} + c.$$

Exercise 2.3.1 Find the following indefinite integrals:

1. $\int \frac{2x+1}{x^2+x}\,dx$

2. $\int \frac{x}{4-x^2}\,dx$

3. $\int \frac{4x^2}{x^3+1}\,dx$

4. $\int \frac{4x}{x^2-1}\,dx$

5. $\int (\frac{1}{x^3} + \frac{1}{x})\,dx$

2.4 Integration Involving Trigonometry

Definition:

$$\int \sin x\,dx = -\cos x + c; \qquad \int \sin kx\,dx = -\frac{1}{k}\cos kx + c$$

$$\int \cos x\,dx = \sin x + c; \qquad \int \cos kx\,dx = \frac{1}{k}\sin kx + c$$

$$\int \sec^2 x\,dx = \tan x + c; \qquad \int \sec^2 kx\,dx = \frac{1}{k}\tan kx + c$$

Example 2.4.1

1. $\int \cos(2x+1)\,dx$

Solution:

$$\int \cos(2x+1)\,dx = \frac{1}{2}\times\sin(2x+1)+c$$
$$= \frac{1}{2}\sin(2x+1)+c.$$

2. $\int \sec^2(3x+1)\,dx$

Solution:

$$\int \sec^2(3x+1)\,dx = \frac{1}{3}\times\tan(3x+1)+c$$
$$= \frac{1}{3}\tan(3x+1)+c.$$

3. $\int \sin 5x\,dx$

Solution:

$$\int \sin 5x\,dx = -\frac{1}{5}\times\cos 5x + c.$$

4. $\int \tan x.\sec^2 x\,dx.$

Solution:

Let $u = \tan x, \Rightarrow$ *then* $du = sec^2 x\,dx$

$$\int \tan x.\sec^2 x\,dx = \int u\,du$$
$$= \frac{1}{2}u^2 + c$$
$$= \frac{1}{2}\tan^2 x + c.$$

2.5 From a Given a Derivative, Find an Integral

Example 2.5.1

1. *Using the product to differentate $x \sin x$.*

> ***Solution:*** *Let $y = x \sin x$. and $u = x$, and $v = \sin x$*
> *By the product rule, $y' = vu' + uv' = \sin x + x.\cos x$.*

2. *Hence find $\int x \cos x dx$.*

> ***Solution:*** *Reversing this result, $\int (\sin x + x.\cos x)dx = x.\sin x$*
>
> $$\int \sin x\, dx + \int x.\cos x\, dx = x.\sin x$$
> $$\therefore \int x.\cos x dx = x.\sin x - \int \sin x dx$$
> $$= x \sin x + \cos x + c.$$

Exercise 2.5.1

1. *Find $\int \cos x.\sin^2 x dx$*

2. $\int \tan^5 x.\sec^2 x\, dx$

2.6 Limits Involving Trigonometric Functions

The Fundamental Limits: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **and** $\lim_{x\to 0} \frac{\tan x}{x} = 1$

Exercise 2.6.1 Find the following limits as $x \to 0$:

1. $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$

2. $\lim_{x\to 0} \frac{2-2\cos 2x}{5x^2}$

3. $\lim_{x\to 0} \frac{1+\cos 2x}{1+\sin 2x}$

4. $\lim_{x\to 0} \frac{\sin 3x}{4x}$

5. $\lim_{x\to 0} \frac{\tan 4x}{\sin \frac{x}{2}}$

2.7 Miscellaneous Exercises

Exercise 2.7.1

1. *Use the substitution* $x = \tan\theta$ *to find the integral* $\int \frac{dx}{\sqrt{x^2+1}}$

2. *Show that* $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ *by multiply top and bottom by* $1+\cos x$

3. *If* $f(x) = kx\tan^{-1} 2x$ *and* $f'(\frac{1}{2}) = \frac{6+3\pi}{4}$, *find the value of k.*

Exercise 2.7.2 Find the following indefinite integrals:

1. $\int \frac{dx}{\sqrt{9-x^2}}$

2. $\int \frac{dx}{\sqrt{9-x}}$

3. $\int \frac{dx}{9-x}$

4. $\int (1-3x)^4 dx$

5. $\int \sin x . \cos^3 x dx$

Exercise 2.7.3 Find the following indefinite integrals:

1. *Find* $\int \frac{2x-1}{x^2+4}\,dx$.

2. *Find* $\int x\sqrt{x^2+9}\,dx$.

3. $\int 4(1-3x)^5\,dx$

4. $\int \frac{e^x}{1+e^{2x}}\,dx$.

2.8 Practical Exam Questions

Exercise 2.8.1 Find the following indefinite integrals:

1. $\int \frac{3}{(x-6)^2}\,dx$

2. $\int \frac{1}{x-5}\,dx$

3. $\int (3x^2 + cos\,2x)\,dx$

4. $\int sec^2\,5x\,dx$

5. $\int \frac{x^2}{x^3+3}\,dx$

Exercise 2.8.2 Find the following indefinite integrals:

1. $\int \frac{6x^2}{x^3-2}\,dx$

2. $\int \frac{x}{x^2+3}\,dx$

3. $\int \frac{x}{x^2-5}\,dx$

4. $\int (4x^3 - 6x + 8)\,dx$

5. $\int (x^2 + \sqrt{x})\,dx$

Exercise 2.8.3

1. $\int \frac{x\,dx}{9+x^2}$

2. *Show that* $\frac{3x-7}{x-2} = 3 - \frac{1}{x-2}$. *Hence, find* $\int \frac{3x-7}{x-2}\,dx$.

3. *Use the substitution* $x = 5\tan\theta$, *find* $\int \frac{x}{(25+x^2)^{\frac{3}{2}}}\,dx$.

4. *Use the substitution* $u = 2x$ *to find* $\int \frac{1}{9+4x^2}\,dx$.