2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
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Table of contents

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10 The Binomial Theorem Part 2

10.1 The Binomial Theorem

The Binomial Theorem: For all cardinal numbers n,

$$^nC_r = \frac{n!}{r!(n-r)!}, \text{ for } r = 0,\ 1,\ 2\ldots,\ n$$

$$\text{Alternatively, } ^nC_r = \frac{n \times (n-1) \times (n-2) \times \ \ldots \times (n-r+1)}{1 \times 2 \times 3 \times \ \ldots \times r}$$

The formula for ^nC_r remains unchanged when r is replace by $n-r$:

$$^nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} =^n C_r$$

Example 10.1.1

1. *Find the general term in the expansion of* $(2x^2 - x^{-1})^{20}$.

Solution:

$$\begin{aligned} \textit{General term} &= {}^{20}C_r \times (2x^2)^{20-r} \times (-x^{-1})^r \\ &= {}^{20}C_r \times 2^{20-r} \times x^{40-2r} \times (-1)^r \times x^{-r} \\ &= {}^{20}C_r \times 2^{20-r} \times (-1)^r \times x^{40-3r}. \end{aligned}$$

2. *Find the term in* x^{34}.

Solution:

To obtain the term in x^{34}, $40 - 3r = 34. \ \Rightarrow \ r = 2.$

$$\begin{aligned} \textit{Hence the term in } x^{34} &= {}^{20}C_2 \times (2x^2)^{20-2} \times (-x^{-1})^2 \\ &= \frac{20 \times 19}{1 \times 2} \times 2^{18} \times x^{36} \times x^{-2} \\ &= 2^{19} \times 5 \times 19 \times x^{34} \\ &= 49807360x^{34}. \end{aligned}$$

3. *Find the term in* x^{-5}.

Solution: *To obtain the term in* x^{-5}, $40 - 3r = -5, \ \Rightarrow \ r = 15.$

$$\begin{aligned} \textit{Hence the term in } x^{-5} &= {}^{20}C_{15} \times (2x^2)^{20-15} \times (-x^{-1})^{15} \\ &= -\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} \times 2^5 \times x^{10} \times x^{15} \\ &= -19 \times 3 \times 17 \times 16 \times 2^5 \times x^{-5} \\ &= -496128x^{-5}. \end{aligned}$$

Some Particular Values of ^nC_r: For all cardinals n,

$$^nC_0 = {^nC_n} = 1, \ \ ^nC_1 = {^nC_{n-1}} = n, \ \ ^nC_2 = {^nC_{n-2}} = \frac{1}{2}n(n-1).$$

Example 10.1.2 Find the value of n if:

1. $^nC_2 = 55$

Solution: *We know* $^nC_0 = {^nC_n} = 1, \ ^nC_1 = {^nC_{n-1}} = n, \ ^nC_2 = {^nC_{n-2}} = \frac{1}{2}n(n-1)$

$$\therefore \ \frac{1}{2}n(n-1) = 55$$
$$n^2 - n - 110 = 0$$
$$(n-11)(n+10) = 0 \ \ \textit{since } n \geq 0, \ \Rightarrow \therefore \ n = 11.$$

2. $^nC_2 +^n C_1 +^n C_0 = 29$

Solution:

$$^nC_2 + {^nC_1} + {^nC_0} = 29$$
$$\frac{1}{2}n(n-1) + n + 1 = 29$$
$$n^2 - n + 2n + 2 = 58$$
$$n^2 + n - 56 = 0$$
$$(n-7)(n+8) = 0 \ \textit{since } n \geq 0, \ \Rightarrow \therefore \ n = 7.$$

Exercise 10.1.1 Use the identities: $^nC_r =^n C_{n-r}$ and $^nC_{r-1} + {^nC_r} = {^{n+1}C_r}$ to solve the following equations for n.

1. $^nC_7 + {^nC_8} = {^{11}C_8}$

2. $^{12}C_4 =^{12} C_n$.

Exercise 10.1.2 Find the specified terms in each of the following expansions:

1. *For the* $(2+x)^7$ *: find the term in* x^4.

2. *For* $(x+\frac{1}{2}y)^{14}$*: find the term in* x^5y^9,

3. *For* $(a-b^{\frac{1}{2}})^{20}$ *: find the term in* $a^3b^{\frac{17}{2}}$.

Exercise 10.1.3 Use the particular formulae for ^nC_r **for** $n = 0, 1,$ **and** 2**, solve each of the following equation for** n**:**

1. $^nC_2 + {^6C_2} = {^7C_2}$,

2. $^nC_2 + {^nC_1} = 22 - {^nC_0}$

3. $^nC_3 + {^nC_2} = 8{^nC_1}$

10.2 Greatest Coefficient and Greatest Term

Definition: In a typical binomial expansion like:
$$(1+2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4,$$
the greatest coefficient is the coefficient of x^3, which is 32.

Example 10.2.1 Write the expansion of $(1+2x)^4$ in the form $\sum_{k=0} t_k x^k$. Find the ratio $\frac{t_{k+1}}{t_k}$.

Solution:

Expanding, $(1+2x)^4 = \sum_{k=0}^{4} {}^4C_k (2x)^k = \sum_{k=0}^{4} {}^4C_k 2^k x^k$

So $(1+2x)^4 = \sum_{k=0}^{4} t_k x^k$, *where* $t_k = {}^4C_k 2^k$.

$$\textit{Hence } \frac{t_{k+1}}{t_k} = \frac{{}^4C_{k+1}2^{k+1}}{{}^4C_k 2^k} = \frac{4!}{(k+1)!(3-k)!} \times \frac{k!(4-k)!}{4!} \times 2 = \frac{(4-k)\times 2}{k+1} = \frac{8-2k}{k+1}.$$

To find where the coefficients are increasing, we solve $t_{k+1} > t_k$,

That is $\frac{t_{k+1}}{t_k} > 1$ *(notice that* t_k *is positive).*

From above, $\frac{8-2k}{k+1} > 1 \Rightarrow 8-2k > k+1 \therefore k < 2\frac{1}{3}$.

So $t_{k+1} > t_k$ *for* $k = 0, 1, 2$ *and* $t_{k+1} < t_k$ *for* $k = 3$. *(k must be an integer).*

Hence $t_0 < t_1 < t_2 < t_3 > t_4$

and the greatest coeffcient is $t_3 = {}^4C_3 \times 2^3 = 32$.

Exercise 10.2.1 Let $(2+3x)^{12} = \sum_{k=0}^{12} t_k x^k$, where $t_k = {}^{12}C_k \times 2^{12-k} \times 3^k$ is the coefficient of x^k. Write down expressions for t_k and t_{k+1} and show that $\frac{t_{k+1}}{t_k} = \frac{36-3k}{2k+2}$.

Definition: In a typical binomial expansion like:

$$(1+\tfrac{3}{2}x)^4 = 1 + 6x + 13\tfrac{1}{2}x^2 + 13\tfrac{1}{2}x^3 + 5\tfrac{1}{16}x^4,$$

there are two greatest coefficients: x^2, which is $13\frac{1}{2}$ and x^3 which is $13\frac{1}{2}$.

Example 10.2.2 Write the expansion of $(1+2x)^4$ in the form $\sum_{k=0}^{n} T_k$. Find the ratio $\frac{T_{k+1}}{T_k}$, and hence find the greatest term if $x = \frac{3}{4}$.

Solution:

From above, $(1+2x)^4 = \sum_{k=0}^{4} T_k$, *where* $T_k = {}^4C_k 2^k x^k$.

$$\begin{aligned}
\frac{T_{k+1}}{T_k} &= \frac{{}^4C_{k+1}2^{k+1}x^{k+1}}{{}^4C_k 2^k x^k} \\
&= \frac{(8-2k)x}{k+1}, \text{ using the previous working,} \\
&= \frac{3(8-2k)}{4(k+1)}, \text{ substituting } x = \frac{3}{4} \\
&= \frac{12-3k}{2k+2}, \text{ after cancelling the 2s.}
\end{aligned}$$

To find where the terms are increasing, we solve $T_{k+1} > T_k$,

that is, $\frac{T_{k+1}}{T_k} > 1$ *(again,* T_k *is positive).*

From above, $\frac{12-3k}{2k+2} > 1 \Rightarrow 12 - 3k > 2k + 2, \therefore k < 2.$

So $T_{k+1} > T_k$ *for* $k = 0$ *and* 1, $T_{k+1} = T_k$ *for* $k = 2$, *and* $T_{k+1} < T_k$ *for* $k = 3$.

Hence $T_0 < T_1 < T_2 = T_3 > T_4$,

and the greatest terms are $T_2 = {}^4C_2 \times \left(\frac{3}{2}\right)^2 = 13.5$, *and* $T_3 = {}^4C_3 \times \left(\frac{3}{2}\right)^3 = 13.5$.

Exercise 10.2.2 Let $(7+3x)^{25} = \sum_{k=0}^{25} C_k x^k$. Write down expressions for C_k and C_{k+1}, and show that $\frac{C_{k+1}}{C_k} = \frac{75-3k}{7k+7}$.

10.3 Identities on the Binomial Coefficients

Basic Binomial Expansion:

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^{n-k} y^k$$

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10.3.1 The First Approach - Substitution

Example 10.3.1 Obtain identities by substituting into the basic expansion:

1. $x = 1$ *and* $y = 1$

Solution:

We begin with the expansion: $(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^{n-k} y^k$.

substituting $x = 1$ *and* $y = 1$ *gives* $2^n = \sum_{k=0}^{n} {}^nC_k$,

That is. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.

In the Pascal triangle, this means that the sum of evry row is 2^n.

2. $x = 1$ *and* $y = -1$

Solution:

Substituting $x = 1$ *and* $y = -1$, *gives* $0 = \sum_{k=0}^{n} {}^nC_k (-1)^k$,

That is, ${}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots (-1)^n \, {}^nC_n = 0$.

This means that the alternating sum of every row is zero.

For odd n, this is trivial: $1 - 5 + 10 - 10 + 5 - 1 = 0$,

but for even n: $1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$.

3. $x = 1$ *and* $y = 2$

Solution:

Substituting $x = 1$ *and* $y = 2$, *gives* $3^n = \sum_{k=0}^{n} {}^nC_k 2^k$,

that is, $1 \times {}^nC_0 + 2 \times {}^nC_1 + 2^2 \times {}^nC_2 + \ldots + 2^n \times {}^nC_n = 3^n$.

10.3.2 Second Approach - Differentiation and Integration

Example 10.3.2 Consider the expansion $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$**.**

1. *Differentiate the expansion, then substitute* $x = 1$ *to obtain an identity.*

Solution:

Differentiating, $n(1+x)^{n-1} = \sum_{k=0}^{n} k \times^n C_k \times x^{k-1}.$

Substituting $x = 1,\ n \times 2^{n-1} = \sum_{k=0}^{n} k \times {}^nC_k,$

That is, $0 \times^n C_0 + {}^nC_1 + 2 \times {}^nC_2 + 3 \times {}^nC_3 + \ldots n \times {}^nC_n = 0.$

Taking as an example the row$1,\ 4,\ 6,\ 4,\ 1,$

$0 \times 1 + 1 \times 4 + 2 \times 6 + 3 \times 4 + 4 \times 1 = 32 = 4 \times 2^3.$

2. *Integrate the expansion, then substitute* $x = -1$ *to obtain an identity. Then give an example of each identity on the Pascal triangle.*

Solution:

Integrating, $\dfrac{(1+x)^{n+1}}{n+1} = C + \sum_{k=0}^{n} \dfrac{{}^nC_k x^{k+1}}{k+1}$, *for som constant C.*

To find the constant C of the integration, substitute $x = 0,$

Then $\dfrac{1}{n+1} = C + \sum_{k=0}^{n} 0$

so $C = \dfrac{1}{n+1}$, *and* $\dfrac{(1+x)^{n+1}}{n+1} = \dfrac{1}{n+1} + \sum_{k=0}^{n} \dfrac{{}^nC_k x^{k+1}}{k+1}.$

Substituting $x = -1,\ \Rightarrow\ 0 = \dfrac{1}{n+1} + \sum_{k=0}^{n} \dfrac{{}^nC_k(-1)^{k+1}}{k+1},$

That is, $\dfrac{1}{x+1} - {}^nC_0 + \dfrac{{}^nC_1}{2} - \dfrac{{}^nC_2}{3} + \dfrac{{}^nC_3}{4} - \ldots + \dfrac{(-1)^{n+1}\,{}^nC_n}{n+1} = 0$

$\times(-1) \Rightarrow\ {}^nC_0 - \dfrac{{}^nC_1}{2} + \dfrac{{}^nC_2}{3} - \dfrac{{}^nC_3}{4} + \ldots + \dfrac{(-1)^n\,{}^nC_n}{n+1} = \dfrac{1}{n+1}$

Taking as an example the row $1,\ 4,\ 6,\ 4,\ 1$

$1 \times 1 - \dfrac{1}{2} \times 4 + \dfrac{1}{3} \times 6 - \dfrac{1}{4} \times 4 + \dfrac{1}{5} \times 1 = 1 - 2 + 2 - 1 = \dfrac{1}{5} = \dfrac{1}{5}.$

10.3.3 Third Approach- Equating Coefficients

Example 10.3.3 Taking $\left(x+\frac{1}{x}\right)^n\left(x+\frac{1}{x}\right)^n = \left(x+\frac{1}{x}\right)^2 n$ **and expanding and equating constants, prove that** $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n$**. Then interpret the identity on the Pascal triangle.**

Solution: *The constant term on the RHS is* ${}^{2n}C_n$.

The constant term on the LHS is the sum of the products:

${}^nC_0 \times {}^nC_n + {}^nC_1 \times {}^nC_{n-1} + {}^nC_2 \times {}^nC_{n-2} + \ldots + {}^nC_n \times {}^nC_0$.

and because ${}^nC_{n-k} =^n C_k,$ *by the symmetry of the row, this constant term is*

$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2+, \ldots + ({}^nC_n)^2$.

Equating the two conatant terms,

$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots({}^nC_n)^2 = {}^{2n}C_n,$ *as required.*

This means that if the entires of any rwo are squared and added,

the sum is the middle enty in the row twice as far down.

For example, with the row $1,\ 3,\ 3,\ 1,\ \Rightarrow\ 1^2 + 3^2 + 3^3 + 1^2 = 20 = {}^6C_3$.

and with the row $1,\ 4,\ 6,\ 4,\ 1\ \Rightarrow\ 1^2 + 4^2 + 6^2 + 4^2 + 1^1 = 70 = {}^8C_4$.

Exercise 10.3.1 By considering the expansion of $(1+x)^{2n} = (1+x)^n(1+x)^n$ **in two different ways, show that** $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$**.**

10.4 Practical Exam Questions

Exercise 10.4.1

1. *Assume the identity* $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$. *By differentiating both sides, and making a suitable substitution for x, show that* $n.2^{n-1} = \sum_{k=1}^{n} k.{}^nC_k$.

2. *If* ${}^nC_5 =^n C_6$, *find the value of* n.

3. *Prove that:* ${}^nC_{k-1} + {}^nC_k = {}^{n+1}C_k$

Exercise 10.4.2

1. *Find the term independent of* $\left(3x^3 - \frac{2}{x}\right)^8$.

2. *find the term independent of x in the expansion* $\left(2x - \frac{1}{x^2}\right)^9$.

3. *If* $(1+x)^n = \sum_{k=0}^{n} C_k x^k$, *find the value of* $\sum_{k=1}^{n} k \times C_k$.

Exercise 10.4.3

1. *State the binomial theorem for* $\left(1+\frac{1}{x}\right)^n$ *where n is a positive integer.*

2. *If* k *is a positive integer, show that* $\left(1+\frac{1}{n}\right)^k$ *approaches 1 as* $n \to \infty$.

3. *Show that* $\frac{1}{n!} < \frac{1}{2^{n-1}}$ *for all positive integral* $n \geq 3$.

4. *Use (3) or otherwise deduce that* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = k$ *where* $2 < k < 3$.

Exercise 10.4.4

1. *By expanding* $[x + (1 - x)]^n$*, for all real numbers x and all positive integers n, when n is an even number.*

2. *Deduce that* ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ \ldots + {}^nC_n = 2^n$.

Exercise 10.4.5

1. *Write the binomial expansion of* $(1 + x)^n$.

2. *By differentiating this expansion, or otherwise, prove that* $\sum_{r=1}^{n} r.{}^nC_r.2^{r-1} = n.3^{n-1}$.

Exercise 10.4.6 Consider the binomial expression:

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots {}^nC_n x^n.$$

1. *Prove that* ${}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots {}^nC_{n-1} = 2^{n-1}$.

2. *Prove that* $\sum_{r=1}^{n} r\,{}^nC_r = n \times 2^{n-1}$.

3. *Find an expression for* $\sum_{r=0}^{n} (r+1)\,{}^nC_r$.