2/3 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
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10 The Binomial Theorem Part 1

10.1 The Pascal Triangle

Expansions of $(1 + x)^n$:

- $(1 + x)^0 = 1$
- $(1 + x)^1 = 1 + x$
- $(1 + x)^2 = 1 + 2x + x^2$
- $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$
- $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

The Pascal Triangle and the addition Property: When the coefficients in the expansions of $(1 + x)^n$ are arranged in a table, the result is known as the Pascal triangle.

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Basic Properties of the Pascal Triangle:

1. Each row starts and ends with 1.
2. Each row is reversible.
3. The sum of each row is 2^n.
4. Every number in the triangle apart from the 1s, is the sum of the number directly above,and the number above and to the left. (the addition property)

Example 10.1.1 Use the Pascal triangle to write out the expansions of:

1. $(1-x)^4$

Solution:
$$\begin{aligned}(1-x)^4 &= 1+4(-x)+6(-x)^2+4(-x)^3+(-x)^4\\ &= 1-4x+6x^2-4x^3+x^4.\end{aligned}$$

2. $(1+2x)^5$.

Solution:
$$\begin{aligned}(1+2x)^5 &= 1+5(2x)+10(2x)^2+10(2x)^3+5(2x)^4+(2x)^5\\ &= 1+10x+40x^2+80x^3+80x^4+32x^5.\end{aligned}$$

3. $(1-\frac{2}{3}x)^6$.

Solution: $\left(1-\dfrac{2}{3}x\right)^6$
$$= 1+6\left(-\frac{2}{3}x\right)+15\left(-\frac{2}{3}x\right)^2+20\left(-\frac{2}{3}x\right)^3+15\left(-\frac{2}{3}x\right)^4+6\left(-\frac{2}{3}x\right)^5+\left(-\frac{2}{3}x\right)^6$$
$$= 1-\frac{12}{3}x+\frac{60}{9}x^2-\frac{160}{27}x^3+\frac{240}{81}x^4-\frac{192}{243}x^5+\frac{64}{729}x^6.$$
$$= 1-4x+\frac{20}{3}x^2-\frac{160}{27}x^3+\frac{80}{27}x^4-\frac{64}{81}x^5+\frac{64}{729}x^6.$$

Exercise 10.1.1

1. *Write out the expresion of* $\left(1+\frac{3}{x}\right)^2$*, then write out the first four terms in the expression of* $(1-x)^8$.

2. *Hence find, in the expansion of* $(1+\frac{3}{x})^2(1-x)^8$. *the term in x.*

Example 10.1.2 Find the value of k if,in the expansion of $(1+2kx)^6$:

1. *The terms in* x^3 *and* x^4 *have coefficients in the ratio* $3:2$,

> **Solution:**
> $$(1+2kx)^6 = \ldots + 15(2kx)^2 + 20(2kx)^3 + 15(2kx)^4 + \ldots$$
> $$= \ldots + 60k^2x^2 + 160k^3x^3 + 240k^4x^4 + \ldots$$
> $$\text{Put } \frac{160k^3}{240k^4} = \frac{3}{2} \Rightarrow \frac{2}{3}k = \frac{3}{2}, \Rightarrow \therefore\ k = \frac{4}{9}.$$

2. *The terms in* x^2, x^3 *and* x^4 *have coefficients i arithmetic progression.*

> **Solution:**
> *Put* $240k^4 - 160k^3 = 160k^3 - 60k^2$.
>
> *Then* $240k^4 - 320k^3 + 60k^2 = 0\ 12k^4 - 16k^3 + 3k^2 = 0$.
>
> *Either* $k = 0$, *or* $12k^2 - 16k + 3 = 0$.
>
> *For the quadratic,* $\Delta = 256 - 144 = 112 = 16 \times 7$,
>
> $$\text{Therefore } k = 0 \text{ or } \frac{16 \pm 4\sqrt{7}}{24}.$$
> $$= 0, \text{ or } \frac{1}{6}(4+\sqrt{7}) \text{ or } \frac{1}{6}(4-\sqrt{7}).$$

Exercise 10.1.2 Find the specified term in each of the following expansions:

1. $(1-3x)^6$, (I) *find the term in* x^4, (II) *find the term in* x^5.

2. *For* $(1-\frac{5}{x})^4$: (I) *find the term in* x^{-1}, (II) *find the term in* x^{-3}.

Exercise 10.1.3

1. *Find the coefficient of* x^4 *in expansion of* $(1-x)^4 + (1-x)^5 + (1-x)^6$.

2. *Find the integers* a *and* b *such that:*

   (a) $(1+\sqrt{3})^5 = a + b\sqrt{3}$

   (b) $(1-2\sqrt{3})^6 = a + b\sqrt{3}$

3. *Expand and simplify:*

   (a) $(1+\sqrt{3})^5 + (1-\sqrt{3})^5$,

   (b) $(1+\sqrt{3})^5 - (1-\sqrt{3})^5$.

Exercise 10.1.4

1. *Find the coefficients of* x^4 *and* x^5 *in the expansions of* $(1+kx)^8$. *Hence find k if these coefficients are in the ratio* $1:5$.

2. *Find the coefficients of* x^3 *and* x^4 *in the expansions of* $(1-kx)^6$. *Hence find k if these coefficients are in the ratio* $2:1$.

3. *Find the coefficients of* x^5 *and* x^6 *in the expansions of* $(1-\frac{3}{4}kx)^9$. *Hence find k if these coefficients are equal.*

10.2 Further Work with the Pascal Triangle

The pattern of the indices in the expansion of $(x+y)^n$:

- $(x+y)^1 = x+y$
- $(x+y)^2 = x^2+2xy+y^2$
- $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$
- $(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$
- $(x+y)^n = *x^n + *x^{n-1}y + *x^{n-2}y^2 + \ldots + *x^2y^{n-2} + *xy^{n-1} + *y^n.$
  (where * denotes the different coefficients.)

The terms of $(x+y)^n$:

- the expansion of $(x+y)^n$ has $n+1$ terms, and in each term the indices of x and y are whole number adding to n.
- That is, the expression $(x+y)^n$ is homogeneous of degree n in x and y together, and so also is its expansion.

The Definition of ${}^nC_r$:

- Define the number ${}^nC_r$ to be the coefficient of x^r in the expansion of $(1+x)^n$.
- The symbol is usually read as 'n choose r', and the notation ${}^nC_r$ and $\binom{n}{r}$ are both used.

The expansion of $(1+x)^n$:

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots {}^nC_nx^n, \text{ term in } x^r =^n C_rx^r.$$

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r.$$

The expansion of $(x+y)^n$:

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots {}^nC_ny^n, \text{ term in } x^{n-r}y^r = {}^nC_rx^{n-r}y^r.$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r.$$

Exercise 10.2.1

1. *Write out the expansion of* $(1+x)^2$ *and* $(1+x)^3$ *using* ${}^nC_r$ *notation.*

2. *Hence give the values of* ${}^2C_0, {}^2X_1$, ${}^2C_2$ *and of* ${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$.

Exercise 10.2.2 Use the Pascal triangle to write out the expansions of:

1. $(2-3x)^4$,

2. $(5x+\frac{1}{5}a)^5$.

3. *Use the Pascal triangle to write out the expansion of* $(2x+x^{-2})^6$, *hence find the term in* x^{-3}.

Exercise 10.2.3

1. *Expand* $(2-3x)^7$ *as far as the term in* x^2*, and hence find the term in* x^2 *in the expansion of* $(5+x)(2-3x)^7$.

2. *Expand* $(1+x)^4$*, and hence write down the value of* ${}^4C_0$, ${}^4C_1$, ${}^4C_3$ *and* ${}^4C_4$.

3. *Expand* $(4+x)^5$ *as far as the term of* x^3*, and hence find the coefficient of* x^3 *in the expansion of* $(3-x)(4+x)^5$.

4. *Expand* $(1-2x)^6$ *as far as the term of* x^4*, and hence find the coefficient of* x^4 *in the expansion of* $(1-3x)(1-2x)^6$.

Exercise 10.2.4

1. *Expand and simplify* $(x+y)^5 + (x-y)^5$.

2. *Find the coefficient of* x^5 *in* $(x^2 - 3x + 11)(4 + x^3)^3$.

3. *Find the coefficient of* x^9 *in* $(x+2)^3(x-2)^7$

4. *Find the coefficients of* x *and* x^{-3} *in the expansion of* $(3x - \frac{a}{x})^5$. *Hence find the values of* a *if these coefficients are in the ratio* $2:1$.

Basic properties of the Pascal triangle:

- Each row starts and ends with 1, and that is ${}^nC_0 = {}^nC_n = 1$ for all cardinals n.
- Each row is reversible, that is,
  ${}^nC_r = {}^nC_{n-r}$, for all cardinals n and r with $r \leq n$.
- The sum of each row is 2^n, that is,
  ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$, for all cardinals n.

The Addition Property: If n and r are positive integers with $r \leq n$, then
${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$, for $1 \leq r \leq n$.

10.3 Factorial Notation

Two definitions of $n!$:

1. For each cardinal n, define n! to be the product of all positive integers from n down to 1:
$n! = n \times (n-1) \times (n-2) \times \ \ldots \times 3 \times 2 \times 1.$

2. Define the function n! recursively by $\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, & \text{for } n \geq 1. \end{cases}$

Example 10.3.1 Simplify the following using unrolling techniques:

1. $\frac{12!}{9!}$

Solution: $$\frac{12!}{9!} = \frac{12 \times 11 \times 10 \times 9!}{9!} = 12 \times 11 \times 10 = 1320.$$

2. $\frac{(n+1)!}{(n-2)!}$

Solution: $$\frac{(n+1)!}{(n-2)!} = \frac{(n+1) \times (n) \times (n-1) \times (n-2)!}{(n-2)!} = (n+1)(n-1)n.$$

3. $\frac{n!}{(n-r)!}$

Solution: $$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!}$$
$$= n(n-1)(n-2)\ldots(n-r+1).$$

Exercise 10.3.1

1. Evaluate $\frac{12!}{2! \times 3! \times 4! \times 5!}$.

2. Simplify by unrolling factorials appropriately: $\frac{n!(n-1)!}{(n+1)!}$.

A Lemma about factorials: Let n and r be cardinal numbers, with $1 \leq r \leq n$.
Then $\frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} = \frac{(n+1)!}{(n-r+1)!r!}$

Exercise 10.3.2 Simplify by taking out a common factor:

1. $(n+1)! - n!$

2. $(n+1)! + (n-1)!$

3. $(n+1)! + n! + (n-1)!$

Exercise 10.3.3 Write each expression as a single fraction:

1. $\frac{1}{n!} - \frac{1}{(n-1)!}$

2. $\frac{1}{(n+1)!} - \frac{1}{(n-1)!}$

Exercise 10.3.4 If $f(x) = x^n$, find:

1. $f'(x)$

2. $f''(x)$

3. $f^n(x)$

Exercise 10.3.5 If $f(x) = \frac{1}{x}$, find:

1. $f'(x)$

2. $f''(x)$

3. $f^5(x)$

4. $f^n(x)$

10.4 Practical Exam Questions

Exercise 10.4.1

1. *Find the value of the term independent of x in the expansion* $(2x + \frac{1}{x^2})^6$.

2. *Expand* $(3 - x)^7$ *as far as the term of* x^4, *and hence find the coefficient of* x^4 *in the expansion of* $(1 - x)^2(3 - x)^7$.

Exercise 10.4.2

1. *Prove that* $\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!}$

2. *Write the binomial expansion of* $(1+x)^n$

3. *By differentiating this expansion, or otherwisr, prove that* $\sum_{r-1}^{n} r \times {}^{n}C_r \times 2^{r-1} = n \times 3^{n-1}$

4. *A student is aloowed to choose one or more books from 6 different books. How many selections are possible?*