2/3 Unit Math Homework for Year 12 (Answers)

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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13 Trigonometry 3

13.1 Integrals Involving Trigonometric Functions

Definition: $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$ $\int \cos(ax+1)\, dx = \frac{1}{a}\sin ax + c.$
$\int \sin ax\, dx = -\frac{1}{a}\cos ax + c.$ $\int \sin(ax+1)\, dx = -\frac{1}{a}\cos ax + c$
$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + c$ $\int \sec^2(ax+1)\, dx = \frac{1}{a}\tan ax + c.$

Example 13.1.1

1. *Find the primitive of* $\sin 5x$.

Solution: *From above,* $\int \sin 5x\, dx = -\frac{1}{5}\cos 5x + c.$

2. *Find* $\int sec^2 \frac{x}{5}\, dx$.

Solution: *From above,* $\int \sec^2 \frac{x}{5}\, dx = \frac{1}{\frac{1}{5}}\tan\frac{x}{5} + c = 5\tan\frac{x}{5} + c$

3. *Find the integral of* $\cos(3x+1)$.

Solution: *From above,* $\int \cos(3x+1)\, dx = \frac{1}{3}\sin(3x+1) + c$

Exercise 13.1.1 Find:

1. $\int \sin 4x\, dx$.

Solution: $\int \sin 4x\, dx = -\frac{1}{4}\cos 4x + c$

2. $\int \cos\frac{x}{4}\, dx$.

Solution: $\int \cos\frac{x}{4}\, dx = 4\sin\frac{x}{4} + c.$

Indefinite integral (Primitive Function) :

$$\int (x^n + ax^n + b)\,dx = \frac{x^{n+1}}{n+1} + \frac{ax^{n+1}}{n+1} + bx + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad \text{where } a, b, c \text{ and } n \text{ are constants, } n \neq -1$$

Definite Integral :

$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

Example 13.1.2

1. *Find* $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$.

Solution:
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin 2x\,dx &= \left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} \\ &= \left(-\tfrac{1}{2}\cos 2\times\tfrac{\pi}{2}\right) - \left(-\tfrac{1}{2}\cos 2\times 0\right) \\ &= \left(-\tfrac{1}{2}\times(-1)\right) - \left(-\tfrac{1}{2}\times 1\right) \\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1\end{aligned}$$

2. $\int_0^{\frac{\pi}{3}} \sin(2-x)\,dx$

Solution:
$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sin(2-x)\,dx &= [(-1)\times -\cos(2-x)]_0^{\frac{\pi}{3}} \\ &= \cos(2-\tfrac{\pi}{3}) - \cos 2 \\ &= 0.9955\end{aligned}$$

3. $\int_{0.5}^{1} \sec^2 x\,dx$

Solution:
$$\begin{aligned}\int_{0.5}^{1} \sec^2 x\,dx &= [\tan x]_{0.5}^{1} \\ &= \tan(1) - \tan(0.5) \\ &= 1.011\end{aligned}$$

Exercise 13.1.2 Find $\int_0^{\frac{\pi}{3}} \cos 3x\,dx$

Solution: $\int_0^{\frac{\pi}{3}} \cos 3x\,dx = \left[\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{3}} = \frac{1}{3}\times(\sin(3\times\frac{\pi}{3}) - \sin 0) = 0$

Exercise 13.1.3 Find the integral of the following functions:

1. $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \sec^2 2x\, dx$

Solution:
$$\begin{aligned}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \sec^2 2x\, dx &= \left[\tfrac{1}{2}\tan 2x\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \\ &= \tfrac{1}{2} \times [(\tan 2 \times \tfrac{\pi}{8}) - (\tan 2 \times (-\tfrac{\pi}{8}))] \\ &= \tfrac{1}{2} \times (1 - (-1)) \\ &= 1\end{aligned}$$

2. *Find* $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos 4x\, dx$

Solution:
$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} \cos 4x\, dx &= \tfrac{1}{4} \times [\sin 4x]_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \\ &= \tfrac{1}{4} \times [(\sin \tfrac{\pi}{2}) - (\sin(-\tfrac{\pi}{2}))] \\ &= \tfrac{1}{4} \times (1 - (-1)) \\ &= \tfrac{1}{2}\end{aligned}$$

3. $\int_0^{\frac{\pi}{2}} \cos \frac{x}{2}\, dx$

Solution:
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos \tfrac{x}{2}\, dx &= 2 \times [\sin \tfrac{x}{2}]_0^{\frac{\pi}{2}} \\ &= 2 \times (\sin \tfrac{\pi}{4} - \sin 0) \\ &= 2 \times (\tfrac{\sqrt{2}}{2} - 0) \\ &= \sqrt{2}\end{aligned}$$

4. $\int_0^{\frac{\pi}{4}} (sec^2 x - x)\, dx$

Solution:
$$\begin{aligned}\int_0^{\frac{\pi}{4}} (sec^2 x - x)\, dx &= \left[\tan x - \tfrac{x^2}{2}\right]_0^{\frac{\pi}{4}} \\ &= \left(\tan(\tfrac{\pi}{4}) - \tfrac{(\frac{\pi}{4})^2}{2}\right) - (\tan 0 - 0) \\ &= 1 - \tfrac{\pi^2}{32} - 0 \\ &= (1 - \tfrac{\pi^2}{32})\end{aligned}$$

Exercise 13.1.4 Find $\int sec^2 4x\, dx$**.**

Solution: $\int sec^2 4x\, dx = \frac{1}{4}\tan 4x + c$

Exercise 13.1.5

1. *Differentiate* $y = \cos(3-2x)$.

Solution: $y = \cos(3-2x)$
$\frac{dy}{dx} = -\sin(3-2x) \times (-2)$
$= 2\sin(3-2x)$

2. *Hence find* $\int \sin(3-2x)$.

Solution: *From above:* $\int \sin(3-2x)\, dx = \frac{1}{2}\cos(3-2x) + c$

Exercise 13.1.6

1. *Differentiate* $y = \cos^3 x$.

Solution: $y = \cos 3x$
$\frac{dy}{dx} = 3\cos^2 x(-\sin x)$
$= -3\cos^2 x . \sin x$

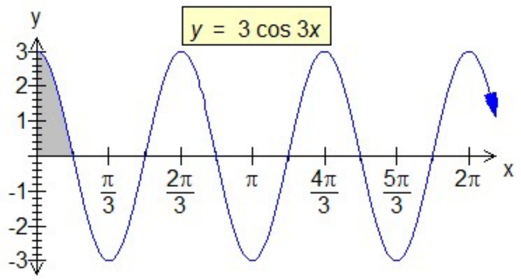
2. *Hence, evaluate* $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx$

Solution: *From above:* $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx = -\frac{1}{3} \times [\cos^3 x]_0^{\frac{\pi}{2}}$
$= -\frac{1}{3}[(\cos\frac{\pi}{2})^3 - (\cos 0)^3]$
$= -\frac{1}{3} \times (-1)$
$= \frac{1}{3}$

13.2 Applications of Integration

Example 13.2.1 Find the area under the curve $= 3\cos 3x$ **between** $x = 0$ **and** $x = \frac{\pi}{6}$**.**

Solution: *First, sketch the curve as shown below: (it's amplitude is 3 and period is* $\frac{2\pi}{3}$*.)*



Area: $A = \int_0^{\frac{\pi}{6}} 3\cos 3x\, dx = 3\left[\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{6}}$

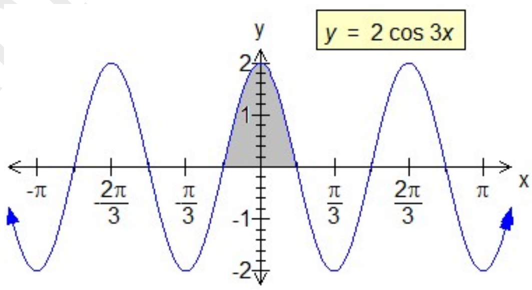
$$= 3\left[\left(\frac{1}{3}\sin 3\times\frac{\pi}{3}\right) - \left(\frac{1}{3}\sin 3\times 0\right)\right]$$

$$= 3\times\left(\frac{1}{3} - 0\right)$$

$$= 1\ \ units^2$$

Exercise 13.2.1 Find the area under the curve $y = 2\cos 3x$**, between** $x = -\frac{\pi}{6}$ **and** $x = \frac{\pi}{6}$**.**

Solution: *First, sketch the curve as shown below: (it's amplitude is 2 and period is* $\frac{2\pi}{3}$*.)*



Area: $A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\cos 3x\, dx = 2\left[\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{3}}$

$$= 2\times\left[\left(\frac{1}{3}\sin 3\times\frac{\pi}{6}\right) - \left(\frac{1}{3}\sin 3\times\left(-\frac{\pi}{6}\right)\right)\right]$$

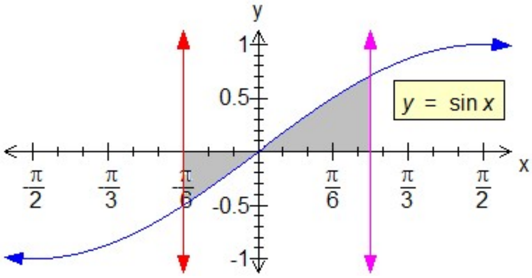
$$= 2\times\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right)$$

$$= \frac{4}{3}$$

$$= 1\frac{1}{3}\ \ units^2$$

Example 13.2.2 Find the area between the curve $y = \sin x$, x-axis and the lines $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

Solution: *First, sketch the curve as shown below:*



The parts of the area above and below the x axis must be found separately and then combined.

Area below the x axis: $A_1 = \left|\int_{-\frac{\pi}{6}}^{0} \sin x\, dx\right|$

$= \left|[-\cos x]_{-\frac{\pi}{6}}^{0}\right|$

$= |(-\cos 0) - (-\cos -(\frac{\pi}{6}))|$

$= |-1 + \frac{\sqrt{3}}{2}|$

$= 1 - \frac{\sqrt{3}}{2}\ units^2$

Area above the x-axis: $A_2 = \int_0^{\frac{\pi}{4}} \sin x\, dx = [-\cos x]_0^{\frac{\pi}{4}}$

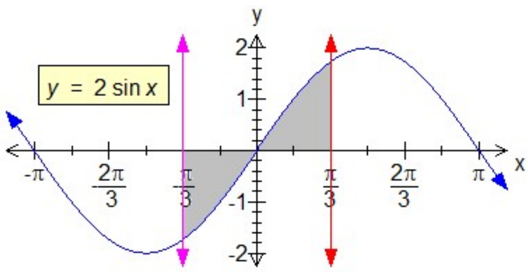
$= (-\cos\frac{\pi}{4}) - (-\cos 0)$

$= -\frac{\sqrt{2}}{2} + 1\, units^2$

$\therefore$ *total area is* $A_1 + A_2 = 1 - \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} + 1 = 2 - \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\ units^2$

Exercise 13.2.2 Find the area enclosed by the curve $y = 2\sin x$, the x-axis, and the lines $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$

Solution: *First, sketch the curve as shown below:*



Area: $A = \left|\int_{-\frac{\pi}{3}}^{0} 2\sin x\, dx\right| + \int_0^{\frac{\pi}{3}} 2\sin x\, dx$

$= \left|2 \times [-\cos x]_{-\frac{\pi}{3}}^{0}\right| + 2 \times [-\cos x]_0^{\frac{\pi}{3}}$

$= \left|2 \times ((-\cos 0) - (-\cos(-\frac{\pi}{3})))\right| + 2 \times \left((-\cos\frac{\pi}{3}) - (-\cos 0)\right)$

$= \left|2 \times (-1 + (\frac{1}{2})\right| + 2 \times (-\frac{1}{2} - (-1))$

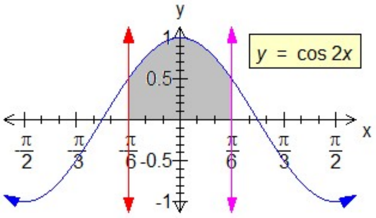
$= 2\, units^2$

13.3 Practical Exam Questions

Exercise 13.3.1

1. *Find the area enclosed by the curve* $y = \cos 2x$*, the axis, and the lines* $x = -\frac{\pi}{6}$ *and* $x = \frac{\pi}{6}$.

Solution: *First, sketch the curve as shown below:*



Area: $A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos x\, dx = \left[\frac{1}{2}\sin 2x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$

$= \frac{1}{2} \times [(\sin 2 \times \frac{\pi}{6}) - (\sin 2 \times (-\frac{\pi}{6}))]$

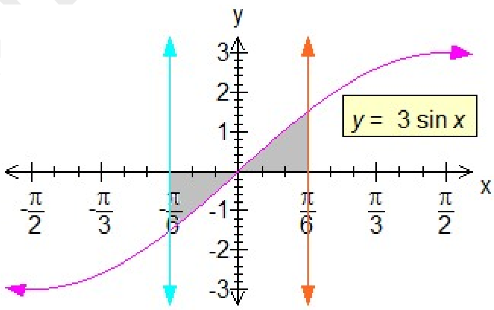
$= \frac{1}{2} \times [(\sin \frac{\pi}{3}) - (\sin \times(-\frac{\pi}{3}))]$

$= \frac{1}{2} \times (\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2})$

$= \frac{\sqrt{3}}{2}\, units^2$

2. *Find the area enclosed by the curve* $y = 3\sin x$*, the axis, and the lines* $x = -\frac{\pi}{6}$ *and* $x = \frac{\pi}{6}$.

Solution: *First, sketch the curve as shown below:*



Area: $A = \left|\int_{-\frac{\pi}{6}}^{0} 3\sin x\, dx\right| + \int_{0}^{\frac{\pi}{6}} 3\sin x\, dx$

$= \left|3 \times [-\cos x]_{-\frac{\pi}{6}}^{0}\right| + 3 \times [-\cos x]_{0}^{\frac{\pi}{6}}$

$= |3 \times [(-\cos 0) - (-\cos(-\frac{\pi}{6}))]| + 3 \times [(-\cos \frac{\pi}{6}) - (-\cos 0)]$

$= \left|3 \times (-1 + \frac{\sqrt{3}}{2})\right| + 3 \times (-\frac{\sqrt{3}}{2} + 1)$

$= 3 - \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} + 3$

$= 6 - 3\sqrt{3}\, units^2$