2/3 Unit Math Homework for Year 12 (Answers)

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

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12 Trigonometry 2

12.1 The Derivative of Trigonometric Functions

Definition: $\frac{d}{dx}(\sin x) = \cos x$ $\qquad \frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$

$\frac{d}{dx}(\cos x) = -\sin x$ $\qquad \frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad \frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$

Example 12.1.1 Find the derivative of :

1. $y = \tan(5x-2)$

Solution: *Let* $u = 5x-2$, *so* $\frac{du}{dx} = 5$, *and* $y = \tan u$, *so* $\frac{dy}{du} = \sec^2 u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \sec^2 u \times 5\\ &= 5\sec^2(5x-2)\end{aligned}$$

2. $y = \frac{e^x}{\cos x}$

Solution: *Let* $u = e^x$, *so* $\frac{du}{dx} = e^x$, *and* $v = \cos x$, *so* $\frac{dv}{dx} = -\sin x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}\\ &= \frac{\cos x.e^x - e^x.-\sin x}{\cos^2 x}\\ &= \frac{e^x(\cos x + \sin x)}{\cos^2 x}\end{aligned}$$

3. $y = \ln(\tan x)$

Solution: *Let* $u = \tan x$, *so* $\frac{du}{dx} = \sec^2 x$ *and* $y = \ln u$, *so* $\frac{dy}{dx} = \frac{1}{u}$

$$\begin{aligned}\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} &= \frac{1}{u} \times \sec^2 x\\ &= \frac{1}{\tan x} \times \sec^2 x\\ &= \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}\\ &= \frac{1}{\cos x \sin x}\\ &= \sec x \, cosec\, x\end{aligned}$$

Example 12.1.2

1. $y = \sin(e^x)$

Solution: *Let* $u = e^x$, *so* $\frac{du}{dx} = e^x$, *and* $y = \sin u$, *so* $\frac{dy}{dx} = \cos u$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u . e^x = e^x \cos(e^x)$$

2. $y = \cos^3 4x$

Solution: *Put* $y = \cos^3 4x = (\cos 4x)^3$, *let* $u = \cos 4x$, *so* $\frac{du}{dx} = -4\sin 4x$

and $y = u^3$, *so* $\frac{dy}{du} = 3u^2$

By the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 3u^2 \times -4\sin 4x$$
$$= -12\sin 4x(\cos 4x)^2$$
$$= -12\sin 4x \cos^2 4x$$

3. $y = 5\cos x - 12\sin x$, *show that* $\frac{d^2y}{dx^2} + y = 0$

Solution: $\frac{dy}{dx} = -5\sin x - 12\cos x$

$$\frac{d^2y}{dx^2} = -5\cos x - 12. - \sin x$$
$$= -5\cos x + 12\sin x$$

So $\frac{d^2y}{dx} + y = (-5\cos x + 12\sin x) + (5\cos x - 12\sin x) = 0$

4. *Find the second derivative of* $\sin^2 x$

Solution: *Put* $y = \sin^2 x = (\sin x)^2 = u^2$, *so* $\frac{dy}{du} = 2u$ *and* $u = \sin x$, $\frac{du}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2\sin x \cos x$$

by the product rule: $\frac{d^2y}{dx^2} = 2\sin x.(-\sin x) + \cos x.2\cos x$

$$= 2(\cos^2 x - \sin^2 x)$$

Exercise 12.1.1 Differentiate the following:

1. $y = \sin^3 2x$

Solution: Put $y = \sin^3 2x = (\sin 2x)^3$, let $u = \sin 2x$, so $\frac{du}{dx} = 2\cos 2x$
and $y = u^3$, so $\frac{dy}{du} = 3u^2$
By the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$= 3u^2 \times 2\cos 2x$
$= 6\sin^2 2x \cos 2x$

2. $y = \cos(e^x)$

Solution: $\frac{dy}{dx} = -e^x \sin(e^x)$

3. $y = \cos(2x - 3)$

Solution: $\frac{dy}{dx} = -2\sin(2x - 3)$

4. $y = x^3 \cos 2x$

Solution: Let $u = x^3$, so $\frac{du}{dx} = 3x^2$, and $v = \cos 2x$, so $\frac{dv}{dx} = -2\sin 2x$
By product rule $\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$
$= x^3.(-2\sin 2x) + \cos 2x.3x^2$
$= -2x^3 \sin 2x + 3x^2 \cos 2x$

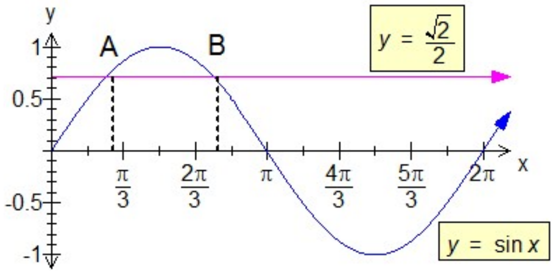
5. $y = \frac{\tan x}{x}$

Solution: Let $u = \tan x$, so $\frac{du}{dx} = \sec^2 x$, $v = x$, so $\frac{dv}{dx} = 1$
$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$
$= \frac{x.\sec^2 x - \tan x.1}{x^2}$
$= \frac{x\sec^2 x - \tan x}{x^2}$

12.2 Using Graphs of Trigonometric Functions

Example 12.2.1 Solve $\sin x = \frac{\sqrt{2}}{2}$, $0 \le x \le 2\pi$

Solution: *(1) Sketch the graphs of* $y = \sin x$ *and* $y = \frac{\sqrt{2}}{2}$*, and note where they intersect.*



(2) Find the first value using your calculator or from the standard values.

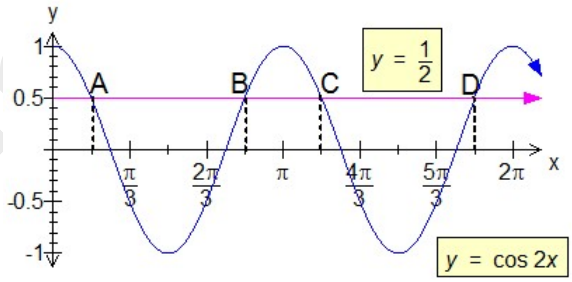
A is where $x = \frac{\pi}{4}$ *or* $(45°)$, $\sin x = \frac{\sqrt{2}}{2}$,

(3) By symmetry, B is where $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$,

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$

Example 12.2.2 Solve $\cos 2x = \frac{1}{2}$, $0 \le x \le 2pi$

Solution: *Sketch the graphs* $y = \cos 2x$ *and* $y = \frac{1}{2}$.



$y = \cos 2x$: $n = 2$, $\therefore$ *Period is* $\frac{2\pi}{2} = \pi$

There are four solutions, from the standard values, if $\cos 2x = \frac{1}{2} : 2x = \frac{\pi}{3}$

$\therefore x = \frac{\pi}{6}$ *(solution A).*

Using symmetry, *B is where* $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

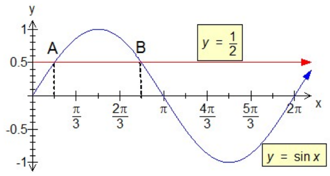
C is where $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

and D is where $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

Exercise 12.2.1 Solve the following equations, where $0 \le x \le 2\pi$

1. $\sin x = \frac{1}{2}$

Solution: *Sketch the graphs of* $y = sinx$*, and* $y = \frac{1}{2}$*, as shown below:*

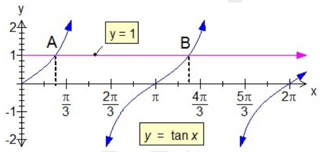


A is where $x = \frac{\pi}{6}$,

By symmetry, B is where $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$*;* $\therefore x = \frac{\pi}{6}, \ \frac{5\pi}{6}$

2. $\tan x = 1$

Solution: *Sketch the graphs of* $y = \tan x$ *and* $y = 1$

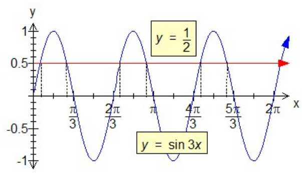


A is where $x = \frac{\pi}{4}$,

By symmetry, B is where $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$*;* $\therefore x = \frac{\pi}{4}, \ \frac{5\pi}{4}$.

3. $\sin 3x = \frac{1}{2}$

Solution: *Sketch the graphs of* $y = \sin 3x$ *and* $y = \frac{1}{2}$



$y = \sin 3x$: $n = 3$ *period is* $\frac{2\pi}{3}$*, there are six solutions:*

The first solution is where $3x = \frac{\pi}{6}, \therefore x = \frac{\pi}{18}$

the second solution is where $x = \frac{\pi}{3} - \frac{\pi}{18} = \frac{5\pi}{18}$

the third solution is where $x = \frac{2\pi}{3} + \frac{\pi}{18} = \frac{13\pi}{18}$

the fourth solution is where $x = \pi - \frac{\pi}{18} = \frac{17\pi}{18}$

the fifth solution is where $x = \frac{4\pi}{3} + \frac{\pi}{18} = \frac{25\pi}{18}$

the sixth solution is where $x = \frac{5\pi}{3} - \frac{\pi}{18} = \frac{29\pi}{18}$

12.3 Applications of the Derivative

Example 12.3.1 Find the gradient of the tangent to the curve $y = 2\sin x$ **at the point where** $x = \frac{\pi}{3}$

Solution: *If* $y = 2\sin x$, $\frac{dy}{dx} = 2\cos x$
$\therefore$ *at* $x = \frac{\pi}{3}$, $\frac{dy}{dx} = 2\cos\frac{\pi}{3}$
$= 2 \times \frac{1}{2}$
$= 1$, $\therefore$ *the gradient of the curve is* $m = 1$

Example 12.3.2 Find in general form the equation of the tangent to the curve $y = \cos 3x$ **at the point where** $x = \frac{\pi}{6}$

Solution: *If* $y = \cos 3x$, $\frac{dy}{dc} = -3\sin 3x$.
At $x = \frac{\pi}{6}$, $\frac{dy}{dx} = -3\sin(3 \times \frac{\pi}{6})$
So $m = -3$.
Also, at $x = \frac{\pi}{6}$, $y = \cos(3 \times \frac{\pi}{6}) = 0$,
So the point is the point $(\frac{\pi}{6},\ 0)$
Now $y - y_1 = m(x - x_1)$
$y - 0 = -3(x - \frac{\pi}{6})$
$y = -3x + \frac{\pi}{2}$.
$\therefore$ *the equation in general form is* $3x + y - \frac{\pi}{2} = 0$

Exercise 12.3.1 Find the equation of the tangent to the curve $y = 2\cos x$ **at the point where** $x = \frac{\pi}{6}$

Solution: *If* $y = 2\cos x$, $\frac{dy}{dx} = -2\sin x$.
At $x = \frac{\pi}{6}$, $\frac{dy}{dx} = -2\sin(\frac{\pi}{6})$,
So $m = -1$
Also, at $x = \frac{\pi}{6}, y = 2\cos(\frac{\pi}{6}) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$
So the point is the point $(\frac{\pi}{6}, \sqrt{3})$
Now $y - y_1 = m(x - x_1)$
$y - \sqrt{3} = -1 \times (x - \frac{\pi}{6})$
$y - \sqrt{3} = -x + \frac{\pi}{6}$
$\therefore x + y - (\sqrt{3} + \frac{\pi}{6})$

Exercise 12.3.2 Find the equation of the tangent to the following curves:

1. $y = 3\sin x$ *at the point where* $x = \frac{\pi}{3}$

Solution: *If* $y = 3\sin x$, $\frac{dy}{dx} = 3\cos x$,
At $x = \frac{\pi}{3}$, $\frac{dy}{dx} = 3\cos\frac{\pi}{3} = \frac{3}{2}$
So $m = \frac{3}{2}$
Also, at $x = \frac{\pi}{3}$, $y = 3\sin(\frac{\pi}{3}) = \frac{3\sqrt{3}}{2}$
Now $y - y_1 = m(x - x_1)$
$y - \frac{3\sqrt{3}}{2} = \frac{3}{2} \times (x - \frac{\pi}{3})$
$\therefore \frac{3}{2}x - y + (\frac{3\sqrt{3}}{2} - \frac{\pi}{2}) = 0$

2. $y = \cos 2x$ *at the point where* $x = \frac{\pi}{4}$

Solution: *If* $y = \cos 2x$, $\frac{dy}{dx} = -2\sin 2x$.
At $x = \frac{\pi}{4}$, $\frac{dy}{dx} = -2\sin(2 \times \frac{\pi}{4})$
So $m = -2$.
Also, at $x = \frac{\pi}{4}$, $y = \cos(2 \times \frac{\pi}{4} = 0)$,
So the point is the point $(\frac{\pi}{4}, 0)$.
Now $y - y_1 = m(x - x_1)$
$y - 0 = -2 \times (x - \frac{\pi}{4})$
$y = -2x + \frac{\pi}{2}$
$\therefore 2x + y - \frac{\pi}{2} = 0$

3. $y = \cos \pi x - 1$ *at the point where* $x = \frac{1}{2}$.

Solution: *If* $y = \cos \pi x - 1$, $\frac{dy}{dx} = -\pi \sin \pi x$,
At $x = \frac{1}{2}$, $\frac{dy}{dx} = -\pi \sin\frac{\pi}{2} = -\pi$
So $m = -\pi$
Also, at $x = \frac{1}{2}$, $y = \cos\frac{\pi}{2} - 1 = -1$,
So the point is the point $(\frac{1}{2}, -1)$.
Now $y - y_1 = m(x - x_1)$
$y - (-1) = -\pi \times (x - \frac{1}{2})$
$\therefore \pi x + y + (1 - \frac{\pi}{2}) = 0$

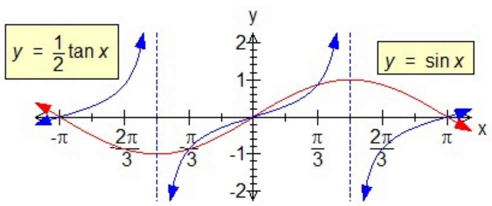
12.4 Practical Exam Questions

Exercise 12.4.1

1. *Show that* $x = \frac{\pi}{3}$ *is a solution of* $\sin x = \frac{1}{2}\tan x$.

> **Solution:** *If* $x = \frac{\pi}{3}$, $LHS = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$,
> $RHS = \frac{1}{2}\tan\frac{\pi}{3} = \frac{1}{2}\times\sqrt{3} = \frac{\sqrt{3}}{2}$, $\therefore x = \frac{\pi}{3}$ *is a solution.*

2. *On the same set of axes, sketch the graphs of the functions* $y = \sin x$ *and* $y = \frac{1}{2}\tan x$ *for* $-\pi \le x \le \pi$.



3. *Hence find all solutions of* $\sin x = \frac{1}{2}\tan x$ *for* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

> **Solution:** *From the graph, solutions are:* $x = -\frac{\pi}{3}, 0$ *and* $\frac{\pi}{3}$

4. *Use your graphs to solve* $\sin x \le \frac{1}{2}\tan x$ *for* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

> **Solution:** *Since* $\sin x \le \frac{1}{2}\tan x$, *and* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
> *When* $\frac{\pi}{3} \le x \le \frac{\pi}{2}$, *and* $-\frac{\pi}{3} \le x \le 0$;

5. *Find the equation of the tangent to* $y = \cos\pi x - 1$ *at the point where* $x = 1$.

> **Solution:** *the gradient of the tangent is* $m = \frac{dy}{dx} = -\pi\sin\pi x$, *when* $x = 1$, $m = \pi\sin\pi = 0$
> *So the gradient of the tangent is parallel to x-axis*
> *and* $y = \cos\pi x - 1$, *when* $x = 1$, $y = -2$.
> $\therefore$ *the equation of the tangent is* $y = -2$

Exercise 12.4.2

1. *Solve* $2\sin^2\frac{x}{3} = 1$ *for* $-\pi \le x \le \pi$.

> ***Solution:*** $2\sin^2\frac{x}{3} = 1$ *for* $-\pi \le x \le \pi$.
> $\therefore \sin^2\frac{x}{3} = \frac{1}{2}$ *for* $-\frac{\pi}{3} \le \frac{x}{3} \le \frac{\pi}{3}$
> $\therefore \sin x = \pm\frac{1}{\sqrt{2}}$
> $\therefore \frac{x}{3} = \frac{\pi}{4},\ \pi - \frac{\pi}{4},\ -\frac{\pi}{4},\ -\pi + \frac{\pi}{4}.$
> $x = \frac{3\pi}{4},\ \frac{9\pi}{4},\ -\frac{3\pi}{4},\ -\frac{9\pi}{4}.$
> $\therefore x = -\frac{3\pi}{4},\ \frac{3\pi}{4}$ *(by domain).*

2. *Solve* $\sqrt{2}\sin x = 1$ *for* $0 \le x \le 2\pi$

> ***Solution:*** $\sqrt{2}\sin x = 1$ *for* $0 \le x \le 2\pi$
> $\sin x = \frac{1}{\sqrt{2}}$
> $\therefore x = \frac{\pi}{4}$ *or* $\pi - \frac{\pi}{4}$
> $\therefore x = \frac{\pi}{4},\ \frac{3\pi}{4}.$

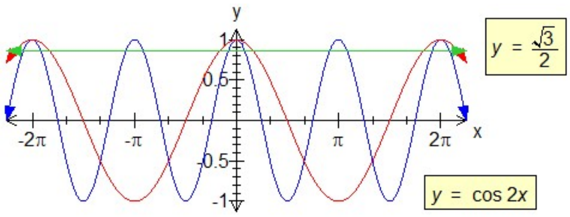
3. *Find the equation of the tangent to the curve* $y = \cos 2x$ *at the point whose x-coordinate is* $\frac{\pi}{6}$.

> ***Solution:*** *If* $y = \cos 2x,\ \frac{dy}{dx} = -2\sin 2x$
> *At* $x = \frac{\pi}{6},\ \frac{dy}{dx} = -2\sin 2(\frac{\pi}{6}) = -2\sin\frac{\pi}{3} = -\sqrt{3}$
> *So the gradient is* $m = -\sqrt{3}$
> *At* $x = \frac{\pi}{6},\ y = \cos 2x = \cos 2(\frac{\pi}{6}) = \cos\frac{\pi}{3} = \frac{1}{2}.$
> *Equation of the tangent at* $(\frac{\pi}{6}, \frac{1}{2})$ *with gradient of* $-\sqrt{3}$ *is:*
> $y - \frac{1}{2} = -\sqrt{3} \times (x - \frac{\pi}{6})$
> $6y - 3 = -6\sqrt{3}x + \sqrt{3}\pi$
> $6\sqrt{3} + 6y - 3 - \sqrt{3}\pi = 0$
> $\therefore\ 6\sqrt{3}x + 6y - (3 + \sqrt{3}\pi) = 0$

12.5 Miscellaneous Exercise

Exercise 12.5.1

1. *Solve the equation* $\cos 2x = \frac{\sqrt{3}}{2}$ *where* $0 \leq x \leq 2\pi$.



Solution: *Sketch the graphs of* $y = \cos 2x$ *and* $y = \frac{\sqrt{3}}{2}$. *The period is* π
We can see that there are four solutions.
From standard values: $\cos 2x = \frac{\sqrt{3}}{2}$, $2x = \frac{\pi}{6}, \therefore x = \frac{\pi}{12}$
Using symmetry, the four solutions are:
$x_1 = \frac{\pi}{12}, x_2 = \pi - \frac{\pi}{12} = \frac{11\pi}{12}, x_3 = \pi + \frac{\pi}{12} = 1\frac{1}{12}\pi$ *and* $x_4 = 2\pi - \frac{\pi}{12} = 1\frac{11}{12}\pi$

2. *Solve the equation* $\sin 3x = -1$, *where* $0 \leq x \leq 2\pi$.

Solution: $\sin 3x = -1, \quad n = 3, \therefore$ *period* $= \frac{2\pi}{3}$
From standard values if $\sin 3x = -1$, $3x = \pi + \frac{\pi}{2}$, *so* $x = \frac{\pi}{2}$
Using symmetry, $x = \frac{2\pi}{3} + \frac{\pi}{2} = \frac{4\pi}{6} + \frac{3\pi}{6} = \frac{7\pi}{6}$
and $x = \frac{4\pi}{3} + \frac{\pi}{2} = \frac{8\pi}{6} + \frac{3\pi}{6} = \frac{11\pi}{6}$ $\qquad \therefore x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3. *If* $f(x) = x - \cos x$, *find* $f'(\frac{\pi}{3})$.

Solution: $f'(x) = 1 - (-\sin x) = 1 + \sin x$
$f'(\frac{\pi}{3}) = 1 + \sin\frac{\pi}{3} = 1 + \frac{\sqrt{3}}{2}$

4. *Differentiate* $y = e^{2x}\sin 2x$

Solution: *Let* $u = e^{2x}$, *so* $\frac{du}{dx} = 2e^{2x}$, *and* $v = \sin 2x$, *so* $\frac{dv}{dx} = 2\cos 2x$
$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{2x}.2\cos 2x + \sin 2x.2e^{2x}$
$= 2e^{2x}(\cos 2x + \sin 2x)$

Exercise 12.5.2

1. *If* $y = \sin x - \cos x$, *show that* $\frac{d^2y}{dx^2} = -y$.

Solution: $y = \sin x - \cos x$, $\frac{dy}{dx} = \cos x + \sin x$
and $\frac{d^2y}{dx^2} = -\sin x + \cos x$
$= -(\sin x - \cos x)$
$= -y$

2. *If* $\frac{dy}{dx} = \cos 3x$, *and* $y = 1$ *when* $x = 0$, *Find* y *when* $x = \frac{\pi}{4}$

Solution: *Since* $\frac{dy}{dx} = \cos 3x$, $y = \frac{1}{3}\sin 3x + c$,
Now $y = 1$ *when* $x = 0$, $1 = \frac{1}{3}\sin(0) + c$, $\Rightarrow c = 1$,
If $x = \frac{\pi}{4}$, $y = \frac{1}{3}\sin\frac{\pi}{4} + 1$
$\therefore\ y = \frac{1}{3} \times \frac{\sqrt{2}}{2} + 1 = \frac{6+\sqrt{2}}{6}$

3. *Defferentiate* $y = e^{3x}\cos 3x$

Solution: *Let* $u = e^{3x}$, $\frac{du}{dx} = 3xe^{3x}$, *and* $v = \cos 3x$, $\frac{dv}{dx} = -3\sin 3x$
$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$= e^{3x}.(-3\sin 3x) + \cos 3x.3xe^{3x}$
$= -3e^{3x}(\sin 3x - x\cos 3x)$

4. *Find the derivative of* $\tan(\ln x)$

Solution: *Let* $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$
and $y = \tan u$, $\frac{dy}{du} = \sec^2 u$
$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ *(Product rule)*
$= \sec^2 u \times \frac{1}{x}$
$= \sec^2(\ln x) \times \frac{1}{x}$
$= \frac{sec^2(\ln x)}{x}$