2/3 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

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12 Trigonometry 2

12.1 The Derivative of Trigonometric Functions

Definition: $\frac{d}{dx}(\sin x) = \cos x$ $\quad \frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$

$\frac{d}{dx}(\cos x) = -\sin x$ $\quad \frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$

Example 12.1.1 Find the derivative of :

1. $y = \tan(5x-2)$

Solution: *Let* $u = 5x-2$, *so* $\frac{du}{dx} = 5$, *and* $y = \tan u$, *so* $\frac{dy}{du} = \sec^2 u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \sec^2 u \times 5\\ &= 5\sec^2(5x-2)\end{aligned}$$

2. $y = \frac{e^x}{\cos x}$

Solution: *Let* $u = e^x$, *so* $\frac{du}{dx} = e^x$, *and* $v = \cos x$, *so* $\frac{dv}{dx} = -\sin x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}\\ &= \frac{\cos x.e^x - e^x.-\sin x}{\cos^2 x}\\ &= \frac{e^x(\cos x+\sin x)}{\cos^2 x}\end{aligned}$$

3. $y = \ln(\tan x)$

Solution: *Let* $u = \tan x$, *so* $\frac{du}{dx} = \sec^2 x$ *and* $y = \ln u$, *so* $\frac{dy}{dx} = \frac{1}{u}$

$$\begin{aligned}\therefore \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} &= \frac{1}{u}\times\sec^2 x\\ &= \frac{1}{\tan x}\times\sec^2 x\\ &= \frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}\\ &= \frac{1}{\cos x\sin x}\\ &= \sec x\, cosec\, x\end{aligned}$$

Example 12.1.2

1. $y = \sin(e^x)$

Solution: *Let* $u = e^x$, *so* $\frac{du}{dx} = e^x$, *and* $y = \sin u$, *so* $\frac{dy}{dx} = \cos u$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \cos u . e^x$$
$$= e^x \cos(e^x)$$

2. $y = \cos^3 4x$

Solution: *Put* $y = \cos^3 4x = (\cos 4x)^3$, *let* $u = \cos 4x$, *so* $\frac{du}{dx} = -4\sin 4x$

and $y = u^3$, *so* $\frac{dy}{du} = 3u^2$

By the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$$= 3u^2 \times -4\sin 4x$$
$$= -12\sin 4x(\cos 4x)^2$$
$$= -12\sin 4x\cos^2 4x$$

3. $y = 5\cos x - 12\sin x$, *show that* $\frac{d^2y}{dx^2} + y = 0$

Solution: $\frac{dy}{dx} = -5\sin x - 12\cos x$
$$\frac{d^2y}{dx^2} = -5\cos x - 12. - \sin x$$
$$= -5\cos x + 12\sin x$$
So $\frac{d^2y}{dx} + y = (-5\cos x + 12\sin x) + (5\cos x - 12\sin x) = 0$

4. *Find the second derivative of* $\sin^2 x$

Solution: *Put* $y = \sin^2 x = (\sin x)^2 = u^2$, *so* $\frac{dy}{du} = 2u$ *and* $u = \sin x$, $\frac{du}{dx} = \cos x$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2\sin x\cos x$$
by the product rule: $\frac{d^2y}{dx^2} = 2\sin x.(-\sin x) + \cos x.2\cos x$
$$= 2(\cos^2 x - \sin^2 x)$$

Exercise 12.1.1 Differentiate the following:

1. $y = \sin^3 2x$

2. $y = \cos(e^x)$

3. $y = \cos(2x - 3)$

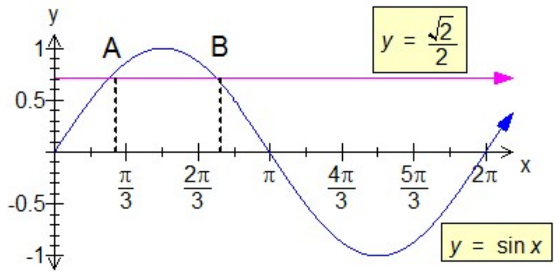
4. $y = x^3 \cos 2x$

5. $y = \frac{\tan x}{x}$

12.2 Using Graphs of Trigonometric Functions

Example 12.2.1 Solve $\sin x = \frac{\sqrt{2}}{2}$, $0 \le x \le 2\pi$

Solution: *(1) Sketch the graphs of* $y = \sin x$ *and* $y = \frac{\sqrt{2}}{2}$*, and note where they intersect.*



(2) Find the first value using your calculator or from the standard values.

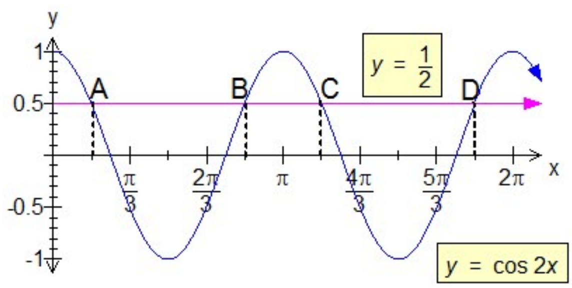
A is where $x = \frac{\pi}{4}$ *or* $(45°)$, $\sin x = \frac{\sqrt{2}}{2}$,

(3) By symmetry, B is where $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$,

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$

Example 12.2.2 Solve $\cos 2x = \frac{1}{2}$, $0 \le x \le 2pi$

Solution: *Sketch the graphs* $y = \cos 2x$ *and* $y = \frac{1}{2}$.



$y = \cos 2x$: $n = 2$, $\therefore$ *Period is* $\frac{2\pi}{2} = \pi$

There are four solutions, from the standard values, if $\cos 2x = \frac{1}{2}$: $2x = \frac{\pi}{3}$

$\therefore x = \frac{\pi}{6}$ *(solution A).*

Using symmetry, *B is where* $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

C is where $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

and D is where $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

Exercise 12.2.1 Solve the following equations, where $0 \leq x \leq 2\pi$

1. $\sin x = \frac{1}{2}$

2. $\tan x = 1$

3. $\sin 3x = \frac{1}{2}$

12.3 Applications of the Derivative

Example 12.3.1 Find the gradient of the tangent to the curve $y = 2\sin x$ at the point where $x = \frac{\pi}{3}$

Solution: *If* $y = 2\sin x,\ \frac{dy}{dx} = 2\cos x$

$\therefore$ *at* $x = \frac{\pi}{3}, \frac{dy}{dx} = 2\cos\frac{\pi}{3}$

$= 2 \times \frac{1}{2}$

$= 1,\ \therefore$ *the gradient of the curve is* $m = 1$

Example 12.3.2 Find in general form the equation of the tangent to the curve $y = \cos 3x$ at the point where $x = \frac{\pi}{6}$

Solution: *If* $y = \cos 3x,\ \frac{dy}{dc} = -3\sin 3x.$

At $x = \frac{\pi}{6},\ \frac{dy}{dx} = -3\sin(3 \times \frac{\pi}{6})$

So $m = -3.$

Also, at $x = \frac{\pi}{6},\quad y = \cos(3 \times \frac{\pi}{6}) = 0,$

So the point is the point $(\frac{\pi}{6},\ 0)$

Now $y - y_1 = m(x - x_1)$

$y - 0 = -3(x - \frac{\pi}{6})$

$y = -3x + \frac{\pi}{2}.$

$\therefore$ *the equation in general form is* $3x + y - \frac{\pi}{2} = 0$

Exercise 12.3.1 Find the equation of the tangent to the curve $y = 2\cos x$ at the point where $x = \frac{\pi}{6}$

Exercise 12.3.2 Find the equation of the tangent to the following curves:

1. $y = 3\sin x$ *at the point where* $x = \frac{\pi}{3}$

2. $y = \cos 2x$ *at the point where* $x = \frac{\pi}{4}$

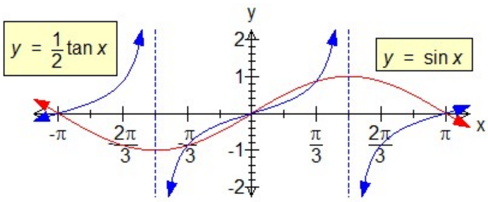
3. $y = \cos \pi x - 1$ *at the point where* $x = \frac{1}{2}$.

12.4 Practical Exam Questions

Exercise 12.4.1

1. *Show that* $x = \frac{\pi}{3}$ *is a solution of* $\sin x = \frac{1}{2}\tan x$.

2. *On the same set of axes, sketch the graphs of the functions* $y = \sin x$ *and* $y = \frac{1}{2}\tan x$ *for* $-\pi \leq x \leq \pi$.



3. *Hence find all solutions of* $\sin x = \frac{1}{2}\tan x$ *for* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

4. *Use your graphs to solve* $\sin x \leq \frac{1}{2}\tan x$ *for* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

5. *Find the equation of the tangent to* $y = \cos \pi x - 1$ *at the point where* $x = 1$.

Exercise 12.4.2

1. *Solve* $2\sin^2\frac{x}{3} = 1$ *for* $-\pi \le x \le \pi$.

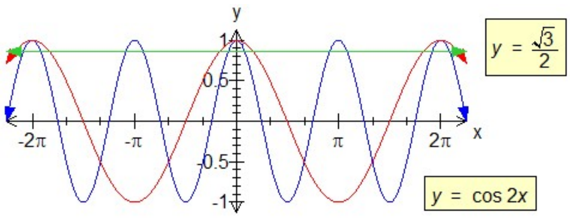
2. *Solve* $\sqrt{2}\sin x = 1$ *for* $0 \le x \le 2\pi$

3. *Find the equation of the tangent to the curve* $y = \cos 2x$ *at the point whose x-coordinate is* $\frac{\pi}{6}$.

12.5 Miscellaneous Exercise

Exercise 12.5.1

1. *Solve the equation* $\cos 2x = \frac{\sqrt{3}}{2}$ *where* $0 \le x \le 2\pi$.



2. *Solve the equation* $\sin 3x = -1$, *where* $0 \le x \le 2\pi$.

3. *If* $f(x) = x - \cos x$, *find* $f'(\frac{\pi}{3})$.

4. *Differentiate* $y = e^{2x} \sin 2x$

Exercise 12.5.2

1. *If* $y = \sin x - \cos x$*, show that* $\frac{d^2y}{dx^2} = -y$.

2. *If* $\frac{dy}{dx} = \cos 3x$*, and* $y = 1$ *when* $x = 0$*, Find* y *when* $x = \frac{\pi}{4}$

3. *Defferentiate* $y = e^{3x} \cos 3x$

4. *Find the derivative of* $\tan(\ln x)$