2/3 Unit Math Homework for Year 12 (Answers)

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

Table of contents

This edition was printed on July 18, 2013.

Camera ready copy was prepared with the LaTeX2e typesetting system.

11 Trigonometry 1

11.1 The Fundamental Identity

Definition: $\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = cosec^2\theta$

Example 11.1.1 Prove the following identities:

1. $\frac{1-\sin^2\theta}{\sin^2\theta+\cos^2\theta} = \cos^2\theta$

Solution:
$$\frac{1-\sin^2\theta}{\sin^2\theta+\cos^2\theta} = \frac{\cos^2\theta}{1} = cos^2\theta$$

2. $\frac{\cos\theta}{1-\sin\theta} - \tan\theta = \sec\theta$

Solution:
$$\begin{aligned}\frac{\cos\theta}{1-\sin\theta} - \tan\theta &= \frac{\cos\theta}{1-\sin\theta} - \frac{\sin\theta}{\cos\theta}\\ &= \frac{\cos^2\theta-\sin\theta(1-\sin\theta)}{(1-\sin\theta)\cos\theta}\\ &= \frac{1-\sin\theta}{(1-\sin\theta)\cos\theta}\\ &= \frac{1}{\cos\theta}\\ &= \sec\theta\end{aligned}$$

3. $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta$

Solution:
$$\begin{aligned}\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta &= \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta\\ &= \sin^2\alpha - \sin^2\alpha\sin^2\beta - (\sin^2\beta - \sin^2\alpha\sin^2\beta)\\ &= \sin^2\alpha - \sin^2\beta\end{aligned}$$

4. $\sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta}$

Solution:
$$\begin{aligned}\sec\theta + \tan\theta &= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\\ &= \frac{1+\sin\theta}{\cos\theta}\end{aligned}$$

Exercise 11.1.1

1. $\frac{\cos\theta\cot\theta}{\cot\theta+\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$

> **Solution:** $$\begin{aligned}\frac{\cos\theta\times\cot\theta}{\cot\theta+\cos\theta} &= \frac{\cos\theta\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}+\frac{\cos^2\theta}{\sin\theta}} \\ &= \frac{\cos^2\theta}{\sin\theta}\times\frac{\sin\theta}{\cos\theta(1+\sin\theta)} \\ &= \frac{\cos\theta}{1+\sin\theta}\end{aligned}$$

2. $\frac{1+\cot\theta}{cosec\,\theta} - \frac{\sec\theta}{\tan\theta+\cot\theta} = \cos\theta$

> **Solution:** $$\begin{aligned}\frac{1+\cot\theta}{cosec\,\theta} - \frac{\sec\theta}{\tan\theta+\cot\theta} &= \frac{1+\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} - \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}} \\ &= \frac{\sin\theta+\cos\theta}{\sin\theta}\times\sin\theta - \frac{1}{\cos\theta}\times\frac{\cos\theta\sin\theta}{\sin^2\theta+\cos^2\theta} \\ &= \sin\theta+\cos\theta-\sin\theta = \cos\theta\end{aligned}$$

3. $\frac{1-\cos^2\theta}{\cos\theta} = \sin\theta\tan\theta$

> **Solution:** $$\begin{aligned}\frac{1-\cos^2\theta}{\cos\theta} &= \frac{\sin^2\theta}{\cos\theta} \\ &= \sin\theta\times\frac{\sin\theta}{\cos\theta} = \sin\theta\tan\theta\end{aligned}$$

4. $1+\frac{1}{\cot^2\theta} = \frac{1}{\cos^2\theta}$

> **Solution:** $$\begin{aligned}1+\frac{1}{\cot^2\theta} &= 1+\frac{1}{\frac{\cos^2\theta}{\sin\theta^2\theta}} \\ &= 1+\frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}\end{aligned}$$

5. $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

> **Solution:** $$\begin{aligned}\frac{1-\cos\theta}{\sin\theta} &= \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \\ &= \frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin\theta}{1+\cos\theta}\end{aligned}$$

11.2 Symmetry properties of trigonometric Functions

Symmetry properties

Degrees	Radians
$\sin(90^\circ - \theta) = \cos\theta$	$\sin(\frac{\pi}{2} - \theta) = \cos\theta$
$\cos(90^\circ - \theta) = \sin\theta$	$\cos(\frac{\pi}{2} - \theta) = \sin\theta$
$\tan(90^\circ - \theta) = \cot\theta$	$\tan(\frac{\pi}{2} - \theta) = \cot\theta$
$\sin(180^\circ - \theta) = \sin\theta$	$\sin(\pi - \theta) = \sin\theta$
$\cos(180^\circ - \theta) = -\cos\theta$	$\cos(\pi - \theta) = -\cos\theta$
$\tan(180^\circ - \theta) = -\tan\theta$	$\tan(\pi - \theta) = -\tan\theta$
$\sin(180^\circ + \theta) = -\sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(180^\circ + \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\tan(180^\circ + \theta) = \tan\theta$	$\tan(\pi + \theta) = \tan\theta$
$\sin(360^\circ - \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(360^\circ - \theta) = \cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(360^\circ - \theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$
$\sin(360^\circ + \theta) = \sin\theta$	$\sin(2\pi + \theta) = \sin\theta$
$\cos(360^\circ + \theta) = \cos\theta$	$\cos(2\pi + \theta) = \cos\theta$
$\tan(360^\circ + \theta) = \tan\theta$	$\tan(2\pi + \theta) = \tan\theta$

Some exact values

	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0
tan	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1	–
cosec	–	2	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	1
sec	1	$\frac{2}{\sqrt{3}}$	2	$\sqrt{2}$	–
cot	–	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	1	0

Example 11.2.1 Find all values of θ between 0 and 2π for which:

1. $\sin\theta = -\frac{1}{\sqrt{2}}$

Solution: $\sin\theta = -\frac{1}{\sqrt{2}}$, *Since $\sin\theta$ is negative, θ lies in the 3rd and 4th quadrants.*

$\therefore\ \theta = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$= \frac{5\pi}{4}, \frac{7\pi}{4}$

2. $\cos\theta = \frac{\sqrt{3}}{2}$

Solution: $\cos\theta = \frac{\sqrt{3}}{2}$, *Since $\cos\theta$ is positive, θ lies in the 1st and 4th quadrants.*

$\therefore\ \theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$= \frac{\pi}{6}, \frac{11\pi}{6}$

Exercise 11.2.1

1. $\sec\theta = -\frac{2}{\sqrt{3}}$

Solution: $\sec\theta = -\frac{2}{\sqrt{3}}$, *Since* $\sec\theta$ *is negative,* θ *lies in the 2nd and 3rd quadrants*

$\therefore \cos\theta = -\frac{\sqrt{3}}{2}$

$\therefore \theta = \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}$

$= \frac{5\pi}{6}, \frac{7\pi}{6}$

2. $\sin\theta = \frac{1}{2}$

Solution: $\sin\theta = \frac{1}{2}$, *Since* $\sin\theta$ *is positive,* θ *lies in the first and 2nd quadrants.*

$\therefore \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$= \frac{\pi}{6}, \frac{5\pi}{6}$

3. $\cos\theta = -\frac{1}{2}$

Solution: $\cos\theta = -\frac{1}{2}$, *Since* $\cos\theta$ *is negative,* θ *lies in the 2nd and 3rd quadrants.*

$\therefore \theta = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$

4. $\sin\theta = -1$

Solution: $\sin\theta = -1$, *Since* $\sin\theta$ *is negative,* θ *lies in the 3rd and 4th quadrants.*

$\therefore \theta = \pi + \frac{\pi}{2}, 2\pi - \frac{\pi}{2}$

$= \frac{3\pi}{2}$

5. $\tan\theta = -\sqrt{3}$

Solution: $\tan\theta = -\sqrt{3}$, *Since* $\tan\theta$ *is negative,* θ *lies in the 2nd and 4th quadrants.*

$\therefore \theta = \pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{5\pi}{3}$

11.3 Derivative of $\sin x$, $\cos x$ and $\tan x$

Definition: $\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$

11.4 Derivative of $\sin(ax+b)$, $\cos(ax+b)$ and $\tan(ax+b)$

Definition: $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$
$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$
$\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$

Example 11.4.1 Differentiate the following:

1. $\sin 5x$.

Solution: *let* $y = \sin 5x$, $\frac{dy}{dx} = 5\cos 5x$

2. $\cos^2 x$.

Solution: *Let* $y = \cos^2 x = u^2$, *where* $u = \cos x$
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2u \times -\sin x = -2\cos x \sin x$

3. $\sin(x^2)$.

Solution: *Let* $y = \sin(x^2) = \sin u$, *where* $u = x^2$,
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times 2x = 2x\cos(x^2)$

4. $x^2 \sin x$.

Solution: *Let* $y = x^2 \sin x = uv$, *where* $u = x^2$ *and* $v = \sin x$
$\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$ *(product rule)*
$= \sin x.2x + x^2 \cos x$
$= 2x\sin x + x^2 \cos x$

Exercise 11.4.1 Differentiate the following with respect to x:

1. $x^2 \cos 2x$

Solution: *Let* $y = x^2 \cos 2x = u.v$, *where* ux^2 *and* $v = \cos 2x$,

$$\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx} = \cos 2x.2x + x^2.(-2\sin 2x) = 2x\cos 2x - 2x^2 \sin 2x$$

2. $x(3\cos x + 2\sin x)$.

Solution: *Let* $y = x(3\cos x + 2\sin x) = u.v$, *where* $u = x$ *and* $v = (3\cos x + 2\sin x)$

$$\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx} = (3\cos x + 2\sin x).1 + x(-3\sin x + 2\cos x) = (3 + 2x)\cos x + (2 - 3x)\sin x$$

3. $\cos^2 3x$

Solution: *Let* $y = \cos^2 3x = u^2$, *where* $u = \cos 3x$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2u.(-3\sin 3x) = 2\cos 3x.(-3\sin 3x) = -6\cos 3x \sin 3x$$

4. $\tan(x^2 - 1)$

Solution: *Let* $y = \tan(x^2 - 1)$,

$$\frac{dy}{dx} = \sec^2(x^2 - 1).2x = 2x\sec^2(x^2 - 1)$$

5. $\frac{1}{2}\cos 2x + x\sin x$

Solution: *Let* $y = \frac{1}{2}\cos 2x + x\sin x$,

$$\frac{dy}{dx} = \frac{1}{2}.(-2\sin 2x) + \sin x.1 + x.\cos x = -\sin 2x + \sin x + x\cos x$$

11.5 Practical Exam Questions

Exercise 11.5.1 A particle P is moving along the x axis. Its position at time t seconds is given by $x = 2\sin t - t$, where $t \leq 0$.

1. *Find an expression for velocity of the particle.*

   Solution: $v = \frac{dx}{dt} = \frac{d(2\sin t - t)}{dt} = 2\cos t - 1$

2. *In what direction is the particle moving at t = 0?*

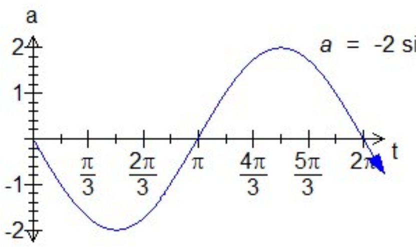
   Solution: *When* $t = 0,\ v = 2\cos(0) - 1 = 2(1) - 1 = +1$ *units/s*
   Hence, the particle is moving away from the origin to the right (positive direction).

3. *Determine when the particle first comes to rest.*

   Solution: *Particle comes to rest when* $v = 0$
   Let $2\cos t - 1 = 0,\ \ 2\cos t = 1,\ \cos t = \frac{1}{2}$
   $\therefore t = \frac{\pi}{3},\ \ t = 2\pi - \frac{\pi}{3}$. *Therefore, particle first comes to rest at* $t = \frac{\pi}{3}$ *seconds.*

4. *When is the acceleration negative for* $0 \leq t \leq 2\pi$*?*

   Solution: $a = \frac{dv}{dt} = -2\sin t,\ \ a < 0$ *when* $0 < t < \pi$



5. *Calculate the total distance travelled by the particle in the first π seconds.*

Solution: *At $t = 0$, particle is moving to the right of 0.*
During the first π seconds, its acceleration is always negative.
The particle will change direction when $v = 0$, i.e. when $t = \frac{\pi}{3}$,
At $t = 0$, $x = 2\sin(0) - 0 = 0$,
At $t = \frac{\pi}{3}$, $x = 2\sin(\frac{\pi}{3}) - \frac{\pi}{3} = 2(\frac{\sqrt{3}}{2}) - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}$
At $t = \pi$, $x = 2\sin(\pi) - \pi = 0 - \pi = -\pi$
$\therefore$ *Total distance travelled* $= (\sqrt{3} - \frac{\pi}{3}) + (\sqrt{3} - \frac{\pi}{3}) + \pi$
$= 2\sqrt{3} - \frac{2\pi}{3} + \pi$
$= (2\sqrt{3} + \frac{\pi}{3})$ *units.*