2/3 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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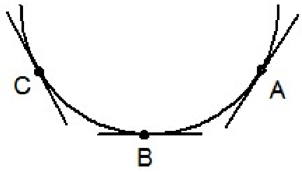
2 Applications of Differentiation

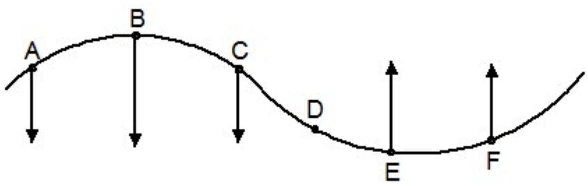
2.1 Significance of First Derivative: Gradient of Tangent



The diagram shown above:

- At point A, $\frac{dy}{dx} > 0$ Curve is increasing at A.
- At point B, $\frac{dy}{dx} = 0$ Curve is stationary at B.
- At point C, $\frac{dy}{dx} < 0$ Curve is decreasing at C.

2.2 Significance of Second Derivative: Concavity



The diagram shown above:

- At points A, B and C $\frac{d^2y}{dx^2} < 0$ Curve is concave downwards.
- At point D $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ change sign passing through D (a point of inflexion).
- At points E and F $\frac{d^2y}{dx^2} > 0$ Curve is concave upwards.

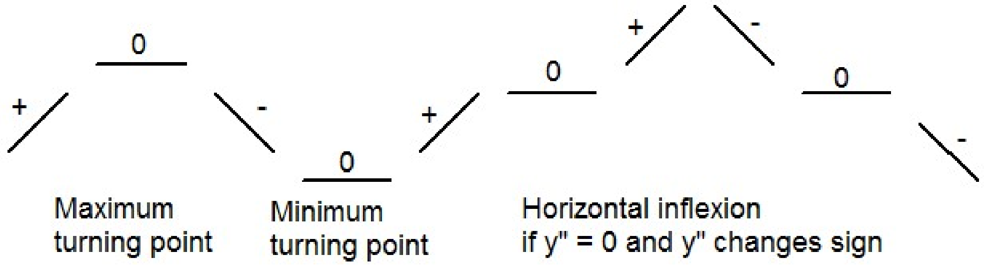
2.3 Stationary Points

- Solve $\frac{dy}{dx} = 0$ for x.
- Determine nature of stationary points (i.e. max or mini point or point of inflexion).
- Find y by substituting x into the original equation.

2.4 Determining Nature of Stationary Points

2.4.1 First Derivative Test (FDT)

Find the gradient of tangent before and after stationary point.



2.4.2 Second Derivative Test (SDT)

Find the concavity at stationary point.

- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ Minimum turning point.
- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ maximum turning point.
- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ changes sign, Horizontal inflexion.

2.5 Point of Inflexion

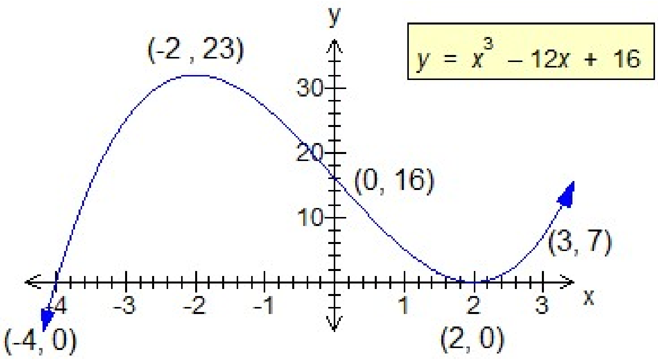
- Solve $\frac{d^2y}{dx^2} = 0$ for x.
- Check that the concavity of the curve changes before and after these values of x.
- Find y by substituting x into the original equation.

2.6 Practical Applications of Differentiation

To find the absolute maximum or minimum value of a quantity, Q:

1. Draw a diagram. Define the variables and label them on the diagram.
2. Find a formula for the quantity, Q being maximised or minimised.
3. Find an equation of the constraint. Hence express Q in terms of one variable, say x.
4. Solve $\frac{dy}{dx} = 0$ for x.
5. Determine whether the value is a relative maximum or minimum using the **First Derivative Test** or **Second Derivative Test**.
6. Examine the values of Q at the end-points of the domain as possible absolute maxima or minima.

Example 2.6.1 Sketch the graph of $f(x) = x^3 - 12x + 16$, in the domain $-4 \le x \le 3$, locating turning points and stating whether they are maximum or minimum points. State the range of $f(x)$ and the coordinates of the extreme points.



Solution:

(Step 1) Solve $\frac{dy}{dx} = 0$ *for x.*

$f(x) = x^3 - 12x + 16$; $f'(x) = 3x^2 - 12$; $3(x-2)(x+2) = 0$; *when* $x = 2$ *or* $x = -2$;

(Step 2) Find y by substituting x into the original equation.

$f(2) = 8 - 24 + 16 = 0$; $f(-2) = -8 + 24 + 16 = 32$

So the coordinates of the stationary points are $(2, 0)$ *and (-2, 32). ;*

(Step 3) Find the gradient of tangent before and after stationary point.

In the neighborhood of $x = -2$, *when* $x < -2$, $f'(x) > 0$,

when $x > -2$, $f'(x) < 0$, *so* $(-2, 32)$ *is a local maximum point.*

In the neighborhood of $x = 2$, *when* $x < 2$, $f'(x) < 0$,

when $x > 2$, $f'(x) < 0$, *so* $(2, 0)$ *is a local minimum point.*

(Step 4) Find $f(x)$ *by substituting x into the original equation.*

$f(0) = 16$ *so the curve crosses the y-axis at* $(0, 16)$,

$f(-4) = 0$ *and* $f(3) = 7$,

The coordinates of the extreme points are $(-4, 0)$ *and* $(3, 7)$,

The range is $0 \le f(x) \le 32$.

Example 2.6.2 The function $f(x) = 3x^2 - x^3$ is defined on the domain $-2 \le x \le 4$.

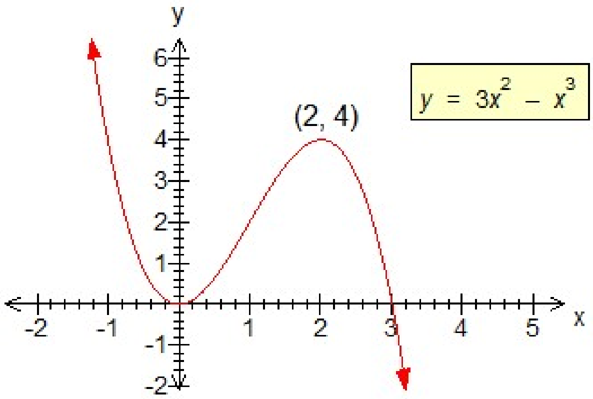
1. *Find the stationary points and their nature.*

Solution:
(Step 1) Solve $\frac{dy}{dx} = 0$ *for x.*
$f(x) = 3x^2 - x^3$, $f'(x) = 6x - 3x^2 = 3x(2 - x)$;
$3x(2 - x) = 0$, *when* $x = 0$ *or* $x = 2$;

(Step 2) Find f(x) by substituting x into the original equation.
When $x = 0,\ f(0) = 0$ *, when* $x = 2,\ f(2) = 4$
The stationary points are (0, 0) and (2, 4);

(Step 3) Find the gradient of tangent before and after stationary point.
When $x < 0,\ f'(x) < 0$ *and when* $x > 0,\ f'(x) > 0$ *so (0, 0) is a local minimum point.*
When $x < 2,\ f'(x) > 0$ *and when* $x > 2,\ f'(x) < 0$ *so (2, 4) is a local maximum point.*

2. *Find the greatest and least values of $f(x)$ in the domain. Hence sketch the graph.*



Solution:
(Step 4) Find $f(x)$ by substituting x into the original equation.
Consider the extreme values of the domain.
When $x = -2,\ y = 20$, *when* $x = 4,\ y = -16$,
The greatest and least value or absolute maximum and minimum are 20 and -16 respectively.

Example 2.6.3 Sketch the graph of $f(x) = 4x^3 - x^4$, locating the turning points and the points of inflexion.

Solution: $f(x) = 4x^3 - x^4$, $f'(x) = 12x^2 - 4x^3$, *and* $f''(x) = 24x - 12x^2$;
For a turning point, $f'(x) = 0$, $\therefore 12x^2 - 4x^3 = 0$, $4x^2(3 - x) = 0$, *So* $x = 0$ *or* 3
When $x = 0$, $f(0) = 4(0)^3 - (0)^4 = 0$, *when* $x = 3$, $f(3) = 4(3^3) - 3^4 = 27$
When $x < 0$, $f'(x) > 0$, *and when* $x > 0$, $f'(x) > 0$,
Hence $f(x)$ *has neither max nor min at* $x = 0$
When $x < 3$, $f'(x) > 0$, *and when* $x > 3$, $f'(x) < 0$,
Hence $f(x)$ *has a local maximum of 27.*

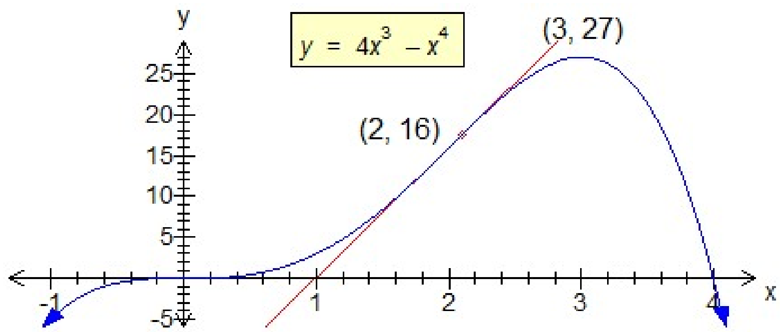
Point of inflexion: $f''(x) = 0$, $f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \therefore x = 0$ *or* 2
When $x = 0$, *both* $f'(x) = 0$ *and* $f''(x) = 0$, *Checking the change in concavity:*

x	*-0.1*	*0*	*0.1*
$f''(x)$	$f''(x) = -2.52 < 0$	$f''(x) = 0$	$= 2.28 > 0$

There is a change in concavity, hence there is a stationary point of inflexion at(0, 0).
When $x = 2$, $f(x) = 16$. *checking the change in concavity:*

x	1.9	2	2.1
$f''(x)$	$f''(x) = 2.28 > 0$	$f''(x) = 0$	$f''(x) = -2.52 < 0$

Hence the point of inflexion at $(2, 16)$ *is a point of maximum positive gradient.*



2.7 Practical Exam Questions

Exercise 2.7.1 Let $f(x) = x^4 - 8x^2$

1. *Find the coordinates of the points where the graph of* $y = f(x)$ *crosses the axes.*

2. *Show that* $f(x)$ *is a even function.*

3. *Find the coordinates of the stationary points of* $f(x)$ *and determine their nature.*

4. *Sketch the graph of* $y = f(x)$.

Exercise 2.7.2 A function is defined by $f(x) = 2x^2(3 - x)$.

1. *Find the coordinates of the turning points of $y = f(x)$ and determine their nature.*

2. *Find the coordinates of point of inflexion.*

3. *What is the minimum value of $f(x)$ for $-1 \leq x \leq 4$?*

4. *Hence sketch the graph of $y = f(x)$, showing the turning points, the point of inflexion and the points where the curve meets the x-axis.*

Exercise 2.7.3 A function is defined by $f(x) = (x + 3)(x^2 - 9)$.

1. *Find all solutions of* $f(x) = 0$.

2. *Find the coordinates of the turning points of the graph of* $y = f(x)$, *and determine their nature.*

3. *Hence sketch the graph of* $y = f(x)$, *showing the turning points and the points where the curve meets the x-axis.*

4. *For what value of x is the graph of* $y = f(x)$ *concave down?*

Exercise 2.7.4 Differentiate with respect to x

1. $f(x) = x\,sin\,x$

2. $f(x) = \frac{x^2}{x-1}$

3. $f(x) = x^2\,log_e\,x$

4. $f(x) = (1 + \sin x)^5$

5. $f(x) = x\,e^{2x}$