4 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
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3 Topic 3 — Polynomials Part 3

3.4 Solutions of Polynomial Equations

Example 3.4.1

1. *Show that i is a zero of* $P(x) = x^3 - 5x + 6i$. *Hence solve* $x^3 - 5x + 6i = 0$.

Solution: $P(i) = -i - 5i + 6i = 0 \ \Rightarrow \ (x - i)$ *is a factor of* $P(x)$.

By inspection, $x^3 - 5x + 6i = (x - i)(x^2 + ix - 6)$

$\therefore P(x) = 0 \ \Rightarrow \ x = i$ *or* $x^2 + ix - 6 = 0,$

Using the quadratic formula, $x = i$ *or* $x = \dfrac{-i \pm \sqrt{23}}{2}$.

2. *Solve* $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$ *over* $\mathbb{C}$. *Hence factorise* $P(x) = x^4 - 4x^3 + 5x^2 - 4x + 1$ *over* $\mathbb{Q}$ *and over* $\mathbb{R}$.

Solution: $P(x) = x^2\left(x^2 - 4x + 5 - \dfrac{4}{x} + \dfrac{1}{x^2}\right) = x^2\left[\left(x^2 + \dfrac{1}{x^2}\right) - 4\left(x + \dfrac{1}{x}\right) + 5\right].$

Using $\left(x + \dfrac{1}{x}\right)^2 = \left(x^2 + \dfrac{1}{x^2}\right) + 2, \Rightarrow \ P(x) = x^2\left[\left(x + \dfrac{1}{x}\right)^2 - 4\left(x + \dfrac{1}{x}\right) + 3\right].$

Since 0 is not a zero of $P(x)$,

the solutions of $P(x) = 0$ *are the solutions of* $\left(x + \dfrac{1}{x}\right)^2 - 4\left(x + \dfrac{1}{x}\right) + 3 = 0.$

by factorising this quadratic $P(x) = x^2(x + \dfrac{1}{x} - 3)(x + \dfrac{1}{x} - 1)$

$= (x^2 - 3x + 1)(x^2 - x + 1)$

$\therefore P(x) = 0 \ \Rightarrow \ x^2 - 3x + 1 = 0$ *or* $x^2 - x + 1 = 0,$

$x = \dfrac{3 \pm \sqrt{5}}{2}$ *or* $x = \dfrac{1 \pm i\sqrt{3}}{2}$.

3. *Show that -i is a zero of* $P(x) = x^3 + ix^2 - 4x - 4i$. *Hence factorise* $P(x)$ *over* $\mathbb{C}$.

Solution: $P(-i) = i - i + 4i - 4i = 0 \ \Rightarrow \ (x + i)$ *is a factor of* $P(x)$.

By inspection, or by polynomial division, $x^3 + ix^2 - 4x - 4i = (x + i)(x^2 - 4)$.

Hence $P(x) = (x + i)(x - 2)(x + 2)$, *and these are irreducible facotr over* $\mathbb{C}$.

Exercise 3.4.1

1. *If* $P(x) = 3x^4 + 10x^3 + 6x^2 + 10x + 3$, *solve* $P(x) = 0$ *over* $\mathbb{C}$ *and factorise* $P(x)$ *over* $\mathbb{R}$.

2. *If* $P(x) = 2x^4 + 7x^3 + 2x^2 - 7x + 2$ *solve* $P(x) = 0$ *over* $\mathbb{C}$, *and factorise* $P(x)$ *over* $\mathbb{R}$.

Definition: Let $z = \cos\theta + i\sin\theta$. then by Demoivre's theorem,

$$z^n = \cos n\theta + i\sin n\theta,\ n = 1,\ 2,\ 3,\ \ldots$$

But by the binomial theorem,

$$z^n = \sum_{k=0}^{n} \binom{n}{k} i^k \sin^k\theta \cos^{n-k}\theta$$

Equating real and imaginary part,

$$\cos n\theta = \cos^n\theta - \binom{n}{2} \sin^2\theta\cos^{n-2}\theta + \binom{n}{4} \sin^4\theta\cos^{n-4}\theta - \ \ldots$$

$$\sin n\theta = (\begin{matrix} n \\ 1 \end{matrix}) \sin\theta\cos^{n-1}\theta - \binom{n}{3} \sin^3\theta\cos^{n-3}\theta + \binom{n}{5} \sin^5\theta\cos^{n-5}\theta - \ \ldots$$

Example 3.4.2 Use De Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **Hence solve** $8x^3 - 6x - 1 = 0$. **Deduce that** $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$.

Solution: *Using the above method for $n = 3$, we obtain by De Moivre's theorem*

$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$

Let $x = \cos\theta$. *Then* $\cos 3\theta = \dfrac{1}{2} \ \Leftrightarrow\ 4x^3 - 3x = \dfrac{1}{2} \ \Rightarrow\ 8x^3 - 6x - 1 = 0.$

Hence if θ is a solution of $\cos 3\theta = \dfrac{1}{2}, \cos\theta$ *is a root of* $8x^3 - 6x - 1 = 0.$

$\cos 3\theta = \dfrac{1}{2} \ \Rightarrow\ 3\theta = 2n\pi \pm \dfrac{\pi}{3},\ $ *n integral,*

$\theta = (6n \pm 1)\dfrac{\pi}{9},\ n = 0,\ \pm1,\ \pm2,\ \ldots$

these values of θ give exactly three distinct values of $\cos\theta$,

namely $\cos\dfrac{\pi}{9},\ \cos\dfrac{5\pi}{9} = -\cos\dfrac{4\pi}{9}$ *and* $\cos\dfrac{7\pi}{9} = -\cos\dfrac{2\pi}{9}.$

Hence the roots of $8x^3 - 6x - 1 = 0 are \cos\dfrac{\pi}{9},\ -\cos\dfrac{4\pi}{9}$ *and* $-\cos\dfrac{2pi}{9}.$

But the coefficient of x^2 is zero and hence the sum of the roots is zero, giving

$\cos\dfrac{\pi}{9} = \cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9}.$

Exercise 3.4.2 Use De Moivre's theorem to show that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$**.**

1. *Find the general solution of tan* $4\theta = 1$.

2. *Hence find the roots of the equation* $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ *in trigonometric form.*

3.5 Partial Fractions

Definition: The highest common factor of two polynomials $A(x)$ and $B(x)$ is the monic polynomial of highest degree which is a factor of both $A(x)$ and $B(x)$. It can be found by repeated use of the division transformation.

Example 3.5.1 Find the highest common factor of $A(x) = x^4 - 3x^3 + 2x^2 - 3x - 9$ **and** $B(x) = x^3 - 4x^2 + 5x - 6$.

Solution: (1) $A(x) \div B(x) \;\Rightarrow$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-4x^2+5x-6 & x^4-3x^3+2x^2-3x-9 \\ & -x^4+4x^3-5x^2+6x \\ \hline & x^3-3x^2+3x-9 \\ & -x^3+4x^2-5x+6 \\ \hline & x^2-2x-3 \end{array}$$

$$\Rightarrow\; x^4-3x^3+2x^2-3x-9 = (x^3-4x^2+5x-6)(x+1)+(x^2-2x-3)$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

$$A(x) \quad = \quad B(x) \quad .Q_1(x) \quad R_1(x)$$

(2) $B(x) \div R_1(x) \;\Rightarrow$

$$\begin{array}{r|l} & x-2 \\ \hline x^2-2x-3 & x^3-4x^2+5x-6 \\ & -x^3+2x^2+3x \\ \hline & -2x^2+8x-6 \\ & 2x^2-4x-6 \\ \hline & 4x-12 \end{array}$$

$$\Rightarrow\; x^3-4x^2+5x-6 = (x^2-2x-30)(x-2)+4(x-3)$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

$$B(x) \quad = \quad R_1(x) \quad .Q_2(x) \quad R_2(x)$$

(3) $R_1(x) \div R_2(x) \;\Rightarrow$

$$\begin{array}{r|l} & \frac{1}{4}x+\frac{1}{4} \\ \hline 4x-12 & x^2-2x-3 \\ & -x^2+3x \\ \hline & x-3 \\ & -x+3 \\ \hline & 0 \end{array}$$

$$\Rightarrow\; x^2-2x-3 = 4(x-3).\frac{1}{4}(x+1)+0$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$R_1(x) = \quad R_2(x) \quad .Q_3(x)$$

The last non-zero remainder $4(x-3)$ gives the highest common factor as $(x-3)$.

Definition: Consider $\frac{P(x)}{Q(x)}$, where $deg\, P < deg\, Q$, and

$Q(x) \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n),\ \alpha_1,\ \alpha_2,\ \ldots\alpha_n$ distinct.

Then we can find a constant c_1 and a polynomial $L_1(x)$ with $deg\, L_1 < deg\, B$.

Such that $\frac{P(x)}{Q(x)} \equiv \frac{c_1}{x-\alpha_1} + \frac{L_1(x)}{B(x)}$. continuing this process

$\frac{P(x)}{Q(x)} \equiv \frac{c_1}{x-\alpha_1} + \frac{c_2}{x-\alpha_2} + \ldots + \frac{c_n}{x-\alpha_n}$,

In practice, $c_1,\ c_2,\ \ldots$ are obtained from the identity by substitution or by equating coefficients in the usual way.

Example 3.5.2 Express $\frac{3x-2}{(x-1)(x-2)}$ as a sum of partial fractions.

Solution: *Let* $\dfrac{3x-2}{(x-1)(x-2)} \equiv \dfrac{c-1}{x-1} + \dfrac{c_2}{x-2}$.

Then $3x-2 \equiv c_1(x-2) + c-2(x-1)$.

Putting $x=1$ *gives* $c_1 = -1$, *while* $x=2$ *gives* $c-2=4$.

Hence $\dfrac{3x-2}{(x-1)(x-2)} \equiv \dfrac{-1}{x-1} + \dfrac{4}{x-2}$

An alternative method of obtaining $c_1,\ c_2\,..c_n$ *in the identity*

$\dfrac{P(x)}{Q(x)} \equiv \dfrac{c_1}{x-\alpha_1} + \dfrac{c_2}{x-\alpha_2} + \ldots + \dfrac{c_n}{x-\alpha_n}$ *where* $Q(x) \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

is to note that $P(x).\dfrac{x-\alpha_1}{Q(x)-Q(\alpha_1)} \equiv c_1 + c_2\dfrac{x-\alpha_1}{x-\alpha_2} + \ldots c_n\dfrac{x-\alpha_1}{x-\alpha_n}$

Since $Q(\alpha_1) = 0. Takinglimitsofbothsidesasx \to \alpha_1, c_1 = \dfrac{P(\alpha_1)}{Q'(\alpha_1)}$.

Similarly, $c_k = \dfrac{P(\alpha_k)}{Q'(\alpha_k)}$, *where* $k = 1,\ 2,\ \ldots n$.

$c_1 = \dfrac{3x-2}{\frac{d}{dx}(x^2-3x+2)}$ *when* $x=1 \;\Rightarrow\; c_1 = -1$.

$c_2 = \dfrac{3x-2}{\frac{d}{dx}(x^2-3x+2)}$ *when* $x=2 \;\Rightarrow\; c_2 = 4$.

Exercise 3.5.1 Express $\frac{x^2+1}{(x+2)(x-1)(x^2+x+1)}$ as a sum of partial fractions over $\mathbb{R}$.

Exercise 3.5.2

1. *Express as a sum of partial fractions* $\frac{2x+10}{(x-1)(x+3)}$.

2. *Express as a sum of partial fractions* $\frac{4x+5}{2x^2+5x+3}$.

3. *Write* $\frac{x+3}{x^4-2x^2+1}$ *as a sum of partial fractions.*

Exercise 3.5.3

1. *Express as a sum of patrials* $\frac{2x+4}{(x-2)(x^2+4)}$.

2. *Express as a sum of partial fractions* $\frac{5-x}{(2x+3)(x^2+1)}$.

3. *Express as a sum of a partial fractions* $\frac{3x^2-3x+2}{(2x-1)(x^2+1)}$.