4 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

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3 Topic 3 — Polynomials Part 2

3.2 Factorisation of polynomials and fundamental theorem of algebra

The division transformation: If $P(x)$ and $D(x)$ are polynomials over a field $\mathbb{F}$, the process of polynomial division of $P(x)$ by $D(x)$ yields a quotient $Q(x)$ and a remainder $R(x)$, which are both polynomials of $\mathbb{F}$, such that $deg\, R < \, deg D$. then:

$$P(x) \equiv D(x)Q(x) + R(x),$$

Example 3.2.1 Consider $P(x) = x^4 - x + 2$ **and** $D(x) = x^2 + 1$**.**

Solution:

$$\begin{array}{r|l} & x^2 \qquad - 1 \\ \hline x^2 + 1 & x^4 \qquad\quad - x + 2 \\ & -x^4 - x^2 \\ \hline & \quad -x^2 - x + 2 \\ & \quad x^2 \qquad + 1 \\ \hline & \qquad\quad -x + 3 \end{array}$$

$$\begin{array}{ccccccc} x^4 - x + 2 \equiv & (x^2+1) & \times & (x^2-1) & + & (-x+3) \\ \uparrow & \uparrow & & \uparrow & & \uparrow \\ P(x) \equiv & D(x) & \times & Q(x) & + & R(x) \end{array}$$

Exercise 3.2.1 Divide $3x^4 - 4x^3 + 4x - 8$ **by** $x^2 - 2$**.**

3.2.1 The Division transformation

The identity $P(x) \equiv D(x).Q(x) + R(x)$ is true for all complex numbers x, when $Q(x)$ and $R(x)$ are polynomials such that $deg\,R < deg\,D$. The nature of the coefficients of $Q(x)$ and $R(x)$ is determined by the nature of he coefficients of $P(x)$ and $D(x)$.

Example 3.2.2 Find the remainder when $P(x) = x^3 - 2x + 1$ is divide by :

1. $x - i$,

Solution: *$x - i$ is a linear divisor.*
Hence we can use the remiander theorem, and the remainder is :
$P(i) = -i - 2i + 1 = 1 + 3i.$

2. $x^2 + 1$.

Solution: *Both $P(x) = x^3 - 2x + 1$ and $D(x) = x^2 + 1$ are polunomials over $\mathbb{Q}$.*
By the division transformation, $P(x) \equiv (x^2 + 1)Q(x) + R(x)$,
where $Q(x)$ and $R(x)$ are polynomials over $\mathbb{Q}$, such that $deg\,R < deg\,D = 2$.
Thus $P(x) \equiv (x^2 + 1)Q(x) + ax + b$, where both a and b are rational,
$\therefore P(i) = 0 + ai + b, \Rightarrow \therefore 1 - 3i = ai + b, \Rightarrow a = -3$ *and* $b = 1$.
Hence the remainder $ax + b$ is $-3x + 1$.

Exercise 3.2.2 If $P(x)$ is divided by $(x - 1)(x + 3)$, remainder is $2x - 11$. What is the remainder when $P(x)$ is divided by $(x - 1)$?

Exercise 3.2.3 When $x^4 - kx + 1$ is divided by $x^2 + 1$, the remainder is $3x + 2$. Find value of k.

Exercise 3.2.4

1. *Find the quotient and remainder when $x^4 - 2x^3 + x^2 - 5x + 7$ is divided by $x^2 + x - 1$.*

2. *Find a and b if $x^4 - 2x^3 + x^2 + ax + b$ is exactly divided by $x^2 + x - 1$.*

3. *Hence factor $x^4 - 2x^3 + x^2 + 8x - 5$.*

Exercise 3.2.5

1. *When* $P(x) = x^4 + ax^2 + 2x$ *is divided by* $x^2 + 1$, *the remainder is* $2x + 3$. *find the value of a.*

2. *When* $P(x) = x^4 + ax^2 + bx + 2$ *is divided by* $x^2 + 1$, *the remainder is* $-x + 1$. *Find the values of* a *and* b.

3.2.2 The Fundamental Theorem of Algebra

3.2.3 Polynomials with Real Coefficients

Exercise 3.2.6

1. *Find the zeros of* $P(x) = x^4 + x^3 - x^2 + x - 2$ *over* $\mathbb{C}$*, given that i is a zero. Hence factor P(x) fully over* $\mathbb{R}$*.*

2. *If* $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$ *has a zero* $1 - i$*, find the zeros of* $P(x)$ *over* $\mathbb{C}$*, and factorise* $P(x)$ *fully over* $\mathbb{R}$*.*

3. $P(x)$ *is an even monic polynomial of degree* 4 *with integer coefficients. One zero is* $2i$ *and the product of the zeros is* -8*. Factorise* $P(x)$ *fully over* $\mathbb{R}$*.*

3.3 The Relationship Between the Roots and Coefficients of a Polynomial

Let $ax^4 + bx^3 + cx^2 + dx + e = 0$ have roots $\alpha, \beta, \gamma, \delta$ over $\mathbb{C}$
Then $ax^4 + bx^3 + cx^2 + dx + e \equiv a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$.

$$\sum \alpha = \frac{b}{a},\ \sum \alpha\beta = \frac{c}{a},\ \sum \alpha\beta\gamma = -\frac{d}{a},\ \alpha\beta\gamma\delta = \frac{e}{a}.$$

For $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$
The sum of the products of roots taken r at a time $= (-1)^r \frac{a_{n-r}}{a_n}$.

Example 3.3.1 Expand $P(x) = (x-1)(x+2)(x-3)(x+1)$

Solution: *$P(x)$ has zeros $1, -1, -2, 3$ and form $x^4 + bx^3 + cx^2 + dx + e$,*
Since $P(x)$ is monic of degree 4. If $\alpha, \beta, \gamma, \delta$ denote the zeros of $P(x)$, then

$\sum \alpha = 1 \ \Rightarrow\ b = -1$

$\sum \alpha\beta = -7 \ \Rightarrow\ c = +(-7)$

$\sum \alpha\beta\gamma = -1 \ \Rightarrow\ d = -(-1)$

$\alpha\beta\gamma\delta = 6 \ \Rightarrow\ e = +6$

Hence $P(x) = x^4 - 3x - 7x^2 + x + 6$.

Exercise 3.3.1

1. *Find the monic polynomial of degree 3 with zeros 1, 2, and 3.*

2. *Two of the roots of* $3x^3 + ax^2 + 23x - 6 = 0$ *are reciprocals. Find the value of a and the three roots.*

Exercise 3.3.2

1. *Two of the roots of* $x^3 - 3x^2 - 4x + a = 0$ *are opposites. Find the value of* a *and the three roots.*

2. *Find the monic polynomial of degrees 4 with zeros* $-3,\ -1,\ 1$ *and* 3.

3. *Two of the zeros of the polynomial* $P(x) = x^4 + bx^3 + cx^2 + dx + e,$ *where* $b,\ c,\ d,$ *and* e *are real, are* $2 + i$ *and* $1 - 3i$. *Find the other two zeros and hence find the values of* b *and* e.

Exercise 3.3.3

1. *The equation* $px^3 + qx^2 + rx + s = 0$ *has roots* $(a - c), a, (a + c)$, *which are in arithmetic progression. Show that the* $a = \frac{-q}{3p}$ *and hence show that* $2q^3 - 9pgr + 27p^2s = 0$.

2. *Solve the equation* $18x^3 + 27x^2 + x - 4 = 0$. *given that the roots are in arithmetic progression.*

3. *The equation* $px^3+qx^2+rx+s = 0$ *has the roots* ac, a *and* $\frac{a}{c}$, *which are in geometric progression. show that* $a = \sqrt[3]{(-\frac{s}{p})}$ *and hence show that* $pr^3 - q^3s = 0$.

Exercise 3.3.4

1. *The equation* $x^3 + 3x^2 - 2x - 2 = 0$ *has roots* $\alpha,\ \beta,$ *and* γ*. Find the equation with the roots (I)* $\alpha - 2,\ \beta - 2$ *and* $\gamma - 2$*; (II)* $\alpha^2,\ \beta^2$ *and* γ^2*.*

2. *The equation* $x^3 + x^2 - 2x - 3 = 0$ *has roots* $\alpha,\ \beta$ *and* γ*. Find the equation s with roots (I)* $\frac{\alpha}{2},\ \frac{\beta}{2}$ *and* $\frac{\gamma}{2}$*; (II)* $\alpha + 2,\ \beta + 2$ *and* $\gamma + 2$*.*

3. *The equation* $x^3 + px + q = 0$ *has roots* $\alpha,\ \beta$ *and* γ*. Find the monic cubic equation with roots* $\alpha^2,\ \beta^2$ *and* γ^2*.*

Exercise 3.3.5

1. *The equation* $x^3 - 6x^2 + ax + 10 = 0$ *has roots that are in arithmetic progression. Find the value of a and solve the equation.*

2. *Solve the equation* $2x63 - 13x^2 - 26x + 16 = 0$*, given that the roots arte in geometric progression.*

3. *The equation* $x^3 + 3x^2 - 2x - 2 = 0$ *has roots* α, β *and* γ*. Find the equations with roots (I)* 2α, 2β *and* 2γ*; (II)* $\frac{1}{\alpha}$, $\frac{1}{\beta}$ *and* $\frac{1}{\gamma}$.