4 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

Table of contents

3 Topic 3 — Polynomials Part 1 **1**

3.1 Factors and zeros of polynomials 1

3.1.1 Multiple zeros of a polynomial 3

This edition was printed on June 16, 2017 with worked solutions.

Camera ready copy was prepared with the LaTeX2e typesetting system.

3 Topic 3 — Polynomials Part 1

3.1 Factors and zeros of polynomials

A polynomial in x over the filed $\mathbb{F}$ has the form:
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0,\ a_0,\dots,a_n \in \mathbb{F},$$

Example 3.1.1 Find the zeros of $P(x) = x^4 - 2x^2 - 3$.

1. *over the field of rational numbers* $\mathbb{Q}$,

Solution: $P(x) = (x^2-3)(x^2+1),\ P(x)$ *has no linear factors over* $\mathbb{Q}$
It cannot be factored further over the field of rational numbers
and we describe these factors as irreducible over $\mathbb{Q}$
Hence $P(x)$ *has no zeros over the field of rational numbers.*

2. *over the field of real numbers* $\mathbb{R}$,

Solution: $P(x) = (x-\sqrt{3})(x+\sqrt{3})(x^2+1).$
Since $P(\sqrt{3}) = P(-\sqrt{3}) = 0.$
Hence the zeros of $P(x)$ *over* $\mathbb{R}$, *are* $\sqrt{3}$ *and* $-\sqrt{3}$.

3. *over complex numbers* $\mathbb{C}$.

Solution: $P(x) = (x-\sqrt{3})(x+\sqrt{3})(x-i)(x+i).$
Hence the zeros of $P(x)$ *over complex number are* $\pm\sqrt{3}$ *and* $\pm i$.

Example 3.1.2 Devide $P(x) = x^2 + x + 3$ **by** $(x-i)$**. Verify that the remainder is** $P(i)$**.**

Solution:

$$\begin{array}{r|lll} & x & +1 & +i \\ \hline x-i) & x^2 & +x & +3 \\ & x^2 & -ix & \\ \hline & & (1+i)x & +3 \\ & & (1+i)x & +(1-i) \\ \hline & & & 2+i \end{array}$$

$\therefore x^2 + x + 3 = (x-i)[x+(1+i)] + (2+i).$

Also $P(i) = x^2 + x + 3 \Rightarrow -1 + i + 3 = 2 + i.$

Definition:

- When $P(x)$ is divided by $(x - \alpha), \alpha \in \mathbb{C}$, the remainder is $P(\alpha)$.
- $(x - \alpha)$ is a factor of $P(x)$ over $\mathbb{C}$ if $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$.

3.1.1 Multiple zeros of a polynomial

Definition:

- If α is a zero of multiplicity r of $P(x)$, $r > 1$, then α is a zero of multiplicity $r-1$ of $P'(x)$.
- Conversely, if $P(x)$ and $P'(x)$ have a common zero α, then α is a multiple zero of $P(x)$.
- If $P(\alpha) = P'(\alpha) = P"(\alpha) = \ldots = P^{r-1}(\alpha) = 0$, and $P^{r}(\alpha) \neq 0$, then α is a zero of multiplicity r of $P(x)$.

Example 3.1.3 Show that $P(x) = x^4 - 2x^3 + 2x - 1$ has a multiple zero. Find this zero and determine its multiplicity. Factor $P(x)$ over R.

Solution:

$P(x) = x^4 - 2x^3 + 2x - 1 \qquad \Rightarrow P(1) = 0$

$P'(x) = 4x^3 - 6x_2^2 \qquad \Rightarrow P'(1) = 0$

$P''(x) = 12x^2 - 12x \qquad \Rightarrow P''(1) = 0$

$P^{(3)}(x) = 24x - 12 \qquad \Rightarrow P^{(3)}(1) \neq 0.$

Hence a is a zero of multiplicity 3 of $P(x)$,

and $P(x) = (x-1)^3(x+k)$ *for some constantk*,

as $P(x)$ *is a monic polynomial of degree* 4.

Then $P(0) = -1 \;\Rightarrow\; k = 1$ *and* $P(x) = (x-1)^3(x+1)$.

Exercise 3.1.1 Show that $\sqrt{7}$ is a root of the equation $2x^3 - 13x - \sqrt{7} = 0$. Hence solve the equation fully over $\mathbb{R}$

Exercise 3.1.2

1. *If* $P(x) = x^3 - x^2 + x - 1$*, divide* $P(x)$ *by* $x + 1$ *and verify that the remainder is* $P(1)$*.*

2. *If* $P(x) = x^3 - 3x^2 + 4x - 2$*, divide* $P(x)$ *by* $(x - i)$ *and verify that the remainder is* $P(-i)$*.*

3. *If* $P(x) = x^3 - 3x^2 + 4x - 2$*, divide* $P(x)$ *by* $(x + 1)$ *and verify that the remainder is* $P(-1)$*.*

Example 3.1.4 $P(x)$ **is a polynomial with integer coefficients.**

1. *Show that if* α *is an integral zero of* $P(x)$*, then* α *is divisor of the constant term.*

Solution: *Let*$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$*, where* $a_0, \ldots a_n$ *are integral.*

$P(\alpha) = \alpha(a_n\alpha^{n-1} + a_{n-1}\alpha^{n-1} + \ldots + a_1) + a_0$

$\therefore P(\alpha) = 0 \Rightarrow a_0 = \alpha(-a_n\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \ldots - a_1)$

But α *integral* $\Rightarrow (-a_n\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \ldots - a_1)$ *integral.*

Hence α *is a divisor of* A_0.

2. *Show that if* $\beta = \frac{p}{q}$ *is rational zero of* $P(x)$*, where* P *and* Q *have no common factor, then P is a divisor of the constant term and q is a divisor of the leading coefficient.*

Solution: $P\left(\frac{p}{q}\right) = a_n\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \ldots + a_1\frac{p}{q} + a_0.$

$\therefore P\left(\frac{p}{q}\right) = 0 \Rightarrow p(a_n p^{n-1} + a_{n-1}qp^{n-2} + \ldots + a_1 q^{n-1}) = -a_0 q^n,$

and $q(a_{n-1}p^{n-1} + a_{n-2}qp^{n-2} + \ldots + a_0 q^{n-1}) = -a_n p^n.$

But, as p *and* q *are integral, each breaketed expression is an integer.*

*Also, since*p *and* q *have no common factor, neither pair of integers* p, q^n

nor p^n, q *has a common factor.*

Hence p *is a divisor of* a_0 *and* q *is a divisor of* a_n.

3. *Show that* $4x^3 + 2x^2 - 14x + 3$ *has exactly one rational zero, and factorise the given polynomial over* $\mathbb{Q}$.

Solution: $P(x) = 4x^3 + 2x^2 - 14x + 3$*.All retional zeros of* $P(x)$ *have the form* $\frac{p}{q}$,

Where p and q are integer divisors of 3 *and* 4 *respectively.*

Hence the only possible retional zero of $P(x)$ *are* $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}.$

But these , only $\frac{3}{2}$ *satisfies* $P(x) = 0$.

Hence $\frac{3}{2}$ *is the only retional zero of* $P(x)$*, and* $(2x - 3)$ *is a factor of* $P(x)$.

By inspection , or by polynomial division,

$4x^3 + 2x^2 - 14x + 3 = (2x - 3)(2x^2 + 4x - 1),$

and these are irreducible factors over $\mathbb{C}$.

Exercise 3.1.3

1. *Express* $P(x) = x^3 + x^2 - 3x - 3$ *as a product of irreducible factors over*
(1) $\mathbb{Q}$; (2) $\mathbb{R}$; (3) $\mathbb{C}$.

2. *Express* $P(x) = x^4 + 3x^3 - 6x^2 - 6x + 8$ *as a product of irreducible factor over*
(1) $\mathbb{Q}$; (2) $\mathbb{R}$; (3) $\mathbb{C}$.

3. *Find* $P(x)$, *given that* $P(x)$ *is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3.*

Exercise 3.1.4

1. *Express* $P(x) = x^3 - 2x^2 + 4x - 8$ *as a product of irreducible factors over* $(i)\ \mathbb{Q},\ (ii)\ \mathbb{R},\ (iii)\ \mathbb{C}$.

2. *Express* $P(x) = x^4 - x^3 - 5x^2 - x - 6$ *as a product of irreducible factors over* $(i)\ \mathbb{Q},\ (ii)\ \mathbb{R},\ (iii)\ \mathbb{C}$.

3. *Divide* $P(x) = x^3 - x^2 + x - 1$ *by* $x - i$ *and verify that the remainder is* $P(i)$.