4 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
Date: ______________	**Score**: ______________

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7 Topic 7 — Mechanics Part 2

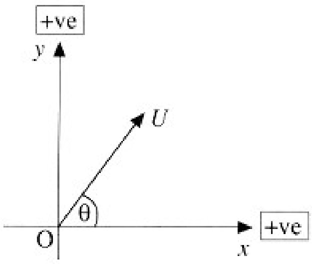
7.3 Motion of a particle in two dimension

7.3.1 Projection Motion

Definition:

- The particle to be moving on a number plane.
- F on the particle is the vector sum of all the physical forces acting on it.
- $\overrightarrow{F} = m\overrightarrow{a}, \Rightarrow F_H = ma_H, F_V = ma_H$.
- where F_H and F_V are the horizontal and vertical components of $\overrightarrow{F}$ respectively.
- $\overrightarrow{F}$ has magnitude mg and is directed vertically downward throughout the motion.

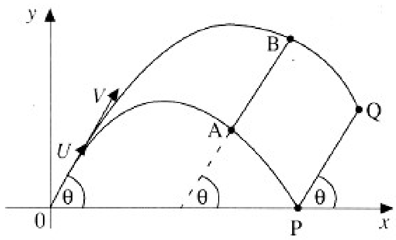
Example 7.3.1 A particle is projected upward over horizontal ground with velocity U inclined at an angle θ to the horizontal. A second particle is projected simultaneously from the same point in the same direction with velocity $V > U$.



1. *Show that through the motion, the line joining the positions of the particles makes a constant angle with the horizontal.*

Solution: *Axes and origin: Initial condition:* $t = 0,\ x = y = 0, \dot{x} = U\cos\theta,\ \dot{y} = U\sin\theta$.
Equation of motion: $\ddot{x} = 0,\ \ddot{y} = -g$
Horizontal component: $\ddot{x} = 0,\ \dot{x} = U\cos\theta,\ \Rightarrow\ x = Ut\cos\theta$
Vertical component: $\ddot{y} = -g,\ \dot{y} = -gt + U\sin\theta,\ \Rightarrow\ y = -\frac{1}{g}t^2 + Ut\sin\theta$
Hence after t seconds, the two particles are at positions:
$A(Ut\cos\theta,\ -\frac{1}{2}gt^2 + Ut\sin\theta)$ *(slower particle)*
$B(Vt\cos\theta,\ -\frac{1}{2}gt^2 + Vt\sin\theta)$ *(faster particle).*
Gradient $AB = \frac{(V-U)t\ \sin\theta}{(V_U)t\ \cos\theta} = \tan\theta$.
Hence AB makes an angle θ with the horizontal throughout the motion.

2. *Given that the range and time of flight of the first particle are respectively* $\frac{1}{g}U^2 \sin 2\theta$ *and* $\frac{2}{g}U \sin\theta$, *find the position of the faster particle and the direction of its velocity vector when the slower particle hits the ground. If this velocity vector is directed upward at an angle* $\beta = \frac{1}{2}\theta$ *to the horizontal. find* $\frac{U}{V}$ *in terms of* θ *and deduce that* $\frac{1}{4} < \frac{U}{V} < \frac{1}{2}$.



Solution: $AB^2 = (V-U)^2t^2 \cos^2\theta + (V-U)^2t^2 \sin^2\theta$

$AB = (V-U)t$ *throughout the motion.*

The slower particle hits the ground at P *when the faster particle is at* Q *and* $t = \frac{2}{g}U\sin\theta$.

Hence QP *makes an angle* θ *with the horizontal and* $QP = (V-U)\frac{2}{g}U\sin\theta$.

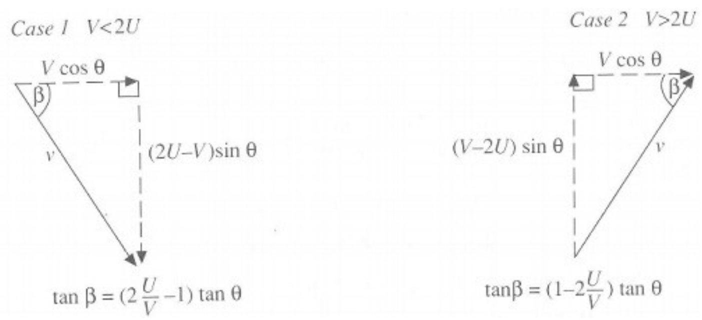
At Q, $x = OP + QP\cos\theta = \frac{1}{g}U^2\sin 2\theta + \frac{1}{g}(V-U)U\sin 2\theta$

$y = QP\sin\theta = \frac{2}{g}U(V-U)\sin^2\theta$

$\therefore$ Q *has coordinates* $(\frac{1}{g}UV\sin 2\theta,\ \frac{2}{g}U(V-U)\sin^2\theta)$.

The components of the velocity vector of the faster particle at Q,

When $t = \frac{2}{g}U\sin\theta$ *are* $\dot{x} = V\cos\theta,\ \dot{y} = (V-2U)\sin\theta$.



Note that if $V = 2U$, *then* $\overrightarrow{v}$ *is horizontal, and the faster particle is at its greatest height.*

If $\beta = \frac{1}{2}\theta$ *and* $\overrightarrow{v}$ *is upward, then* $V > 2U$ *and* $\tan\left(\frac{\theta}{2}\right) = \left(1 - 2\frac{U}{V}\right)\tan\theta$.

Let $\lambda = \tan\frac{\theta}{2}$ *then* $\frac{U}{V} < \frac{1}{2}$ *and* $\lambda = \left(1 - 2\frac{U}{V}\right)\frac{2\lambda}{1-\lambda^2}$

$\therefore 1 - \lambda^2 = \left(2 - 4\frac{U}{V}\right)$

$\lambda^2 = 4\frac{U}{V} - 1$

$\frac{U}{V} = \frac{1}{4}(\lambda^2 + 1)$

$\therefore \frac{U}{V} = \frac{1}{4}\sec^2\left(\frac{\theta}{2}\right)$ *and* $\frac{1}{4} < \frac{U}{V} < \frac{1}{2}$.

Exercise 7.3.1

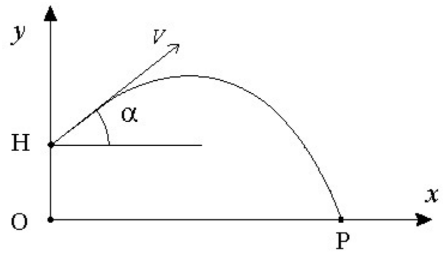
1. *A particle of mass m is projected vertically upward under gravity in a medium in which the resistance is mk times the square of the speed, where k is a positive constant. The speed of projection is V. Find the maximum height.*

2. *A particle of mass $0.5\,kg$ is released from rest and moves vertically downward under gravity in a medium which exerts a resistance to the motion of $\frac{1}{10}v^2$. At time t after release it has fallen a distance x and has velocity v. Taking $g = 10\,ms^{-1}$, show that $v^2 = 50(1 - e^{0.4x})$ and $\ddot{x} = 10e^{-0.4x}$.*

Exercise 7.3.2

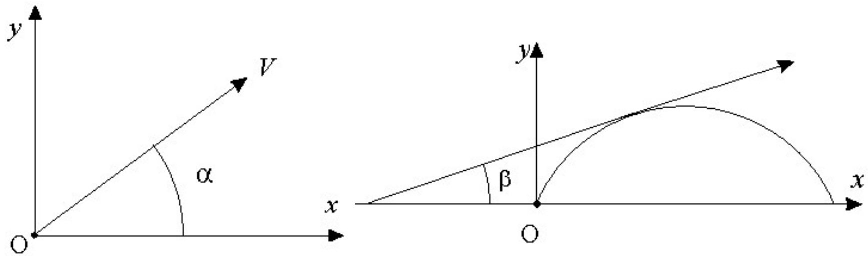
1. *A particle of mass* $1\ kg$ *is projected vertically upward under gravity with speed* $2c$ *in a medium in which the resistance to motion is* $\frac{g}{c^2}$ *times the square of the speed, where* c *is a positive constant. Find the time of ascent of the particle and the maximum height.*

2. *A particle is projected with speed* V *and angle of elevation* α *from a point* O *on the edge of a cliff of height* h. *When the particle hits the ground its path makes an angle* $\tan^{-1}(2\tan\alpha)$ *with the horizontal. Find the distance from the foot of the cliff to the point where it lands.*

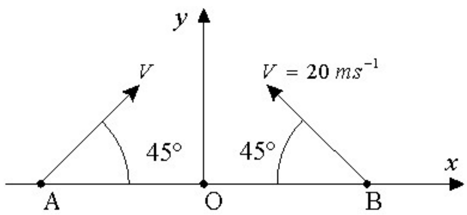


Exercise 7.3.3

1. *A particle is projected from a point O at time $t = 0$ with speed V and angle of elevation α . It moves under gravity and reaches its horizontal range R at time $t = T$. If the direction of motion of the particle makes an angle β with the horizontal when $t = \frac{1}{4}T$, show that $\tan\beta = \frac{1}{2}\tan\alpha$.*

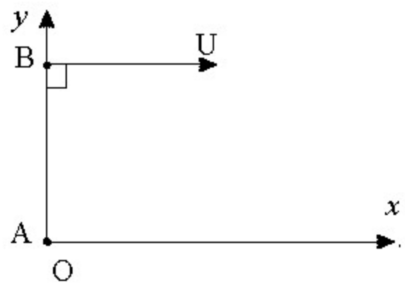


2. *A and B are two points on level ground, $40\,m$ apart. Simultaneously a particle is projected from A towards B and another particle is projected from B towards A, each with speed $20\,ms^{-1}$ at an angle of elevation of 45° . Given that the two particles collide, find the time and the height above AB at which this occurs.*

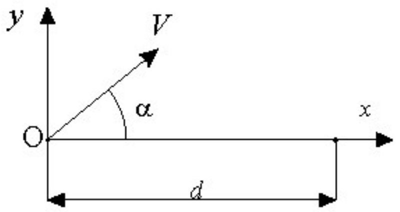


Exercise 7.3.4

1. *A particle is projected under gravity horizontally with the speed* $30\,ms^{-1}$ *from a point* B $45\,m$ *vertically above a point* O *on horizontal ground. Taking* $g = 10\,ms^{-2}$*, find the time taken for the particle to reach the ground and the horizontal distance it has then traveled.*

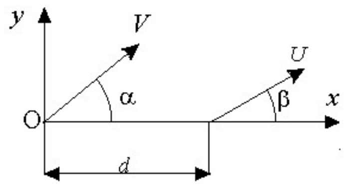


2. *A projectile is fired with speed* V *at an angle of elevation* α *from a point* O *and hits a stationary target at a distance* d *from* O *on the same level. Find the value of* V*.*



Exercise 7.3.5

1. *A projectile is fired with speed V at an angle of elevation α from a point O. At the instant of projection the target is fired from a point at a distance d from O on the same level with speed u and angle of elevation β in the plane of the path of the projectile and away from O. Given that the projectile hits the target, find the time at which this occurs.*



2. *A and B are two points on level ground $110\,m$ apart. A particle is projected from A towards B with speed $60\,ms^{-1}$ at an angle of elevation of $30°$. At the same instant another particle is projected from B towards A with speed $50\,m^{-1}$. Given that the two particles collide, find the angle of projection of the second particle and the time of collision.*

7.4 Circular Motion in a Horizontal Plane

Definition:

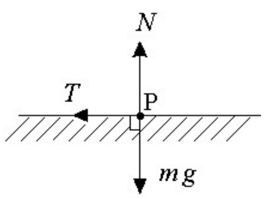
- The linear velocity of P is a vector $\vec{v}$ in the direction of motion and its direction along the tangent at P.
- If ℓ is the arc length from the x-axis to P, then $v = \frac{d\ell}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$
- The linear velocity $\vec{v}$, measured in ms^{-1},
- angular velocity ω, measured in radians s^{-1}.

Tension forces: When particles are connected by a taut string, the string exerts a tension force on each particle.

Reaction forces:

- A body P in contact with a surface exerts a force on the surface and the surface exerts an equal and opposite force on P.
- These forces are termed an action-reaction pair and the force on P by the surface is called a reaction force.
- The reaction force acts at right angles to the surface.

Example 7.4.1 A mass of $2\,kg$ is revolving at the end of a string $2\,m$ long on a smooth horizontal table with uniform angular speed of 1 revolution per second. Find the tension in the string.



Solution: *Observed acceleration is $a = \ell\omega^2$,*
where $\omega = 2\pi\, rad\, s^{-1}$, and $\ell = 2m$ is the length of a string.
The resultant has horizontal component $ma = m\ell\omega^2 \Rightarrow T = m\ell\omega^2$.
$m = 2,\ \ell = 2, \omega = 2\pi, \Rightarrow T = 16\pi^2\, N.$

7.4.1 Uniform circular motion

If P is moving in a circle of radius r with constant speed v and constant angular velocity ω, the motion is described as uniform circular motion and the following results apply:

- Linear velocity is $v = r\omega$, directed along the tangent.
- Linear acceleration is $a = r\omega^2$, directed towards the centre.
- the vector sum of all the physical forces on P (the resultant force) is: $F = mr\omega^2, \Rightarrow F = \frac{mv^2}{r}$ directed towards the centre.
- If f is the number of revolutions per second (frequency) and T is the time taken for one revolution (period), $\omega = 2\pi f = \frac{2\pi}{T}$.

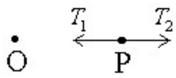
Exercise 7.4.1

1. *A particle of mass $m\,kg$ is travelling at constant speed $v\,ms^{-1}$ round a circle of radius $r\,m$. If $v = 8$ and $r = 2$, find the magnitude of the linear acceleration.*

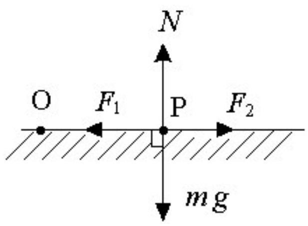
2. *A particle of mass $0.5\,kg$ is attached to one end of a light inextensible string of length $2\,m$. The other end is fixed to a point A on a smooth horizontal table. The particle is set in motion in a circular path. If the speed of the particle is $12\,ms^{-1}$, find the tension in the string.*

Exercise 7.4.2

1. *A particle of mass* $0.1\,kg$ *moving on a smooth horizontal table with constant speed* $v\,ms^{-1}$ *describes a circle with center O and radius* $r\,m$ *. The particle is attracted towards O by a force of magnitude*$4v\,N$ *and repelled from O by a force of magnitude* $\frac{k}{r}\,N$ *where* k *is a constant. Given that* $v = 40$ *and the time of one revolution is* $\frac{\pi}{10}$ *seconds, find the values of* r *and* k*.*



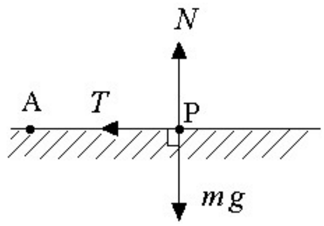
2. *A particle P of mass* $0.2\,kg$ *moving on a smooth horizontal table with constant speed* $v\,ms^{-1}$ *describes a circle with center O such that* $OP = r\,m$ *. The particle is subject to two forces, one towards O with magnitude* $8v\,N$ *and one away from O with magnitude* $\frac{k}{r^2}\,N$*, where* k *is a positive constant. Given that* $k = 75$ *and* $r = 1$ *, find the possible values of* v*.*



Exercise 7.4.3

1. *A particle of mass* $m\,kg$ *is travelling at constant speed* $v\,ms^{-1}$ *round a circle of radius* $r\,m$. *If* $v = 3$, $r = 6$, *and the force acting towards the centre of the circle is of constant magnitude* $6\,N$, *find the value of* m.

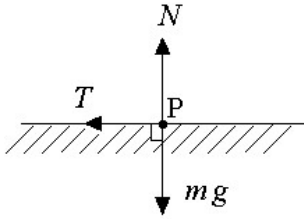
2. *A particle of mass* $0.25\,kg$ *is attached to one end of a light inextensible string of length* $\ell = 0.5\,m$. *The other end is fixed to a point* A *on a smooth horizontal table. The particle is set in motion in a circular path. If the speed of the particle is* $8\,ms^{-1}$ *, find the tension in the string and the reaction with the table.*



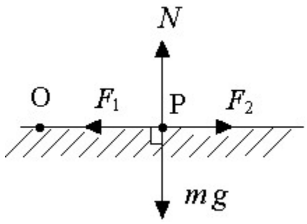
3. *A particle of mass* $0.5\,kg$ *is attached to one end of a light inextensible string of length* $2\,m$. *The other end is fixed to a point A on a smooth horizontal table. The particle is set in motion in a circular path. If the string breaks when the tension in it exceeds* $64\,N$, *find the greatest speed at which the particle can travel.*

Exercise 7.4.4

1. *A mass of* $2\,kg$ *is revolving at the end of a string* $2\,m$ *long on a smooth horizontal table with uniform angular speed. If the string would break under a tension equal to the weight of* $20\,kg$*, find the greatest positive speed of the mass.*

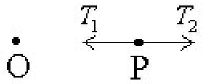


2. *A particle P of mass 0.2 kg moving on a smooth horizontal table with constant speed describes a circle with centre O such that . The particle is subject to two forces, one towards O with magnitude 8v N an one away from O with magnitude N, where k is a positive constant. If , find the set of possible values of k.*

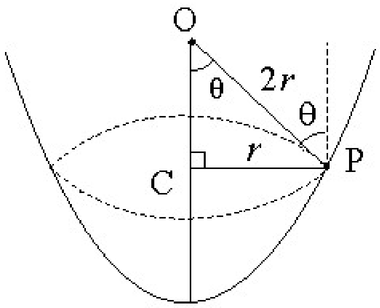


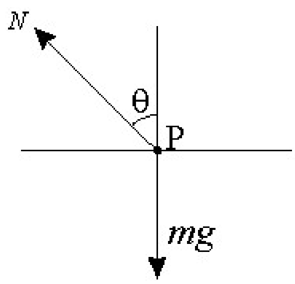
Exercise 7.4.5

1. *A particle of mass* $0.1\,kg$ *moving on a smooth horizontal table with constant speed* $v\,ms^{-1}$ *describes a circle with centre O and radius* $r\,m$*. The particle is attracted towards O by a force of magnitude* $4v\,N$ *and repelled from O by a force of magnitude* $\frac{k}{r}\,N$*, where* k *is a constant. Given that* $k = 30$ *and* $r = 1$*, find the possible values of* v*.*



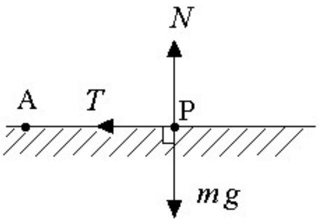
2. *A particle moves with constant angular velocity* ω *in a horizontal circle of radius* r *on the inside of a fixed smooth hemispherical bowl of internal radius* $2r$*. Show that* $\omega^2 = \frac{g}{r\sqrt{3}}$*.*





Exercise 7.4.6

1. *A particle of mass* $0.25\,kg$ *is attached to one end of light inextensible string of length* $0.5\,m$*. The other end is fixed to a point A on a smooth horizontal table. The particle is set in motion in a circular path. If the string breaks when the tension in it exceeds* $50\,N$*, find the greatest angular velocity at which the particle can travel.*



2. *A mass of* $1\,kg$ *is fastened by a string of length* $1\,m$ *to a point* $0.5\,m$ *above a smooth horizontal table and is describing a circle on the table with uniform angular speed of* 1 *revolution in* 2 *seconds. Find the force exerted on the table and the tension in the string.*