4 Unit Math Homework for Year 12

Student Name: ______________________	**Grade**: ______________________
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7 Topic 7 — Mechanics Part 1

7.1 Motion of a Partial in One Dimension

7.1.1 Velocity and Acceleration

Definition: When investigating the motion of a particle in one dimension, we consider the particle to be moving along a number line:

- The instantaneous velocity v of the particle is its rate of change of position $\dot{x} = \frac{dx}{dt}$,
- The instantaneous acceleration is its rate of change of velocity $\ddot{x} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.
- When the equation of motion gives $\ddot{x}$ as a function of v or x, the results $\ddot{x} = \frac{1}{2}\frac{dv^2}{dx}$ or $\ddot{x} = v\frac{dv}{dx}$.

Example 7.1.1 Initially a particle is observed to be travelling in a straight line with speed $V\,ms^{-1}$. Throughout the subsequent motion, the particle is slowing down at a rate proportional to its speed. Find expression for the velocity v of the particle when it is x metres from its initial position, and the velocity of the particle and its displacement from its initial position after t seconds. Show that this displacement has a limiting value at $t \to \infty$.

Solution: *Origin and positive direction: Choose initial position as origin, and initial direction of motion as positive.*

Equation of motion: Let the $\ddot{x} = -kv, \; k > 0$ *constant,*

Initial condition: $t = 0 \;\Rightarrow x = 0$ *and* $v = V\,ms^{-1}$

Relation between x and v: $\ddot{x} = -kv \;\Rightarrow v\frac{dv}{dx} = -kv, \Rightarrow \frac{dv}{dx} = -k$

$\therefore \; v = -kx + c, \;\; c$ *constant*

When $x = 0, \; v = V,$ *and* $c = V \;\; \therefore v = V - kx$ (1)

Relation between v and t: $\ddot{x} = -kv \;\Rightarrow \frac{dv}{dt} = -kv \;\Rightarrow -k\frac{dt}{dv} = \frac{1}{v}$

$\therefore -kt = \ln(Av), \;\; A$ *constant*

When $t = 0, \; v = V \;\Rightarrow AV = 1, \; \therefore -kt = \ln(\frac{v}{V}, \; v = Ve^{-kt}$ (2)

From (1) *and* $(2), V - kx = Ve^{-kt}, \therefore x = \frac{V}{k}(1 - e^{-kt})$ (3)

From $(2), \;\; 0 \le v \le V$ *and* $v \to 0^+$ *as* $t \to \infty$.

From $(3), \;\; 0 \le x < \frac{V}{k}$ *and* $x \to (\frac{V}{k})^-$ *as* $t \to \infty$.

Note that we integrate $\frac{1}{v}$ *with respect to* v*, it is more convenient to incorporate the constant of integration in the argument of the logarithm function.*

In $Av = \ln v + \ln A = \ln v + c$*, where* $c = \ln A$*.*

Hence In Av *A constant, and* $\ln v + c$*,* c *constant, are equivalent expression for the indefinite integral.*

Exercise 7.1.1

1. *A particle movies in a straight line with acceleration which is inversely proportional to* t^3*, where* t *is the time. The particle has a velocity of* $3\,ms^{-1}$ *when* $t = 1$ *and its velocity approaches a limiting values of* $5\,ms^{-1}$*. Find an expression for its velocity at time* t*.*

2. *A particle movies in a straight line from a fixed point O in the line, such that when its distance from O is* x *its speed v is given by* $v = \frac{k}{x}$*, for some constant* k*. Show that the particle has a retardation which is inversely proportional to* x^3*.*

3. *A particle movies in a straight line away from a fixed point O in the line, such that at time* t *its displacement from O is x and its displacement from O is x and its velocity is* v*. At time* $t = 0, x = 0$*, and* $v = V$*. Subsequently the particle is slowing down at a rate proportional to the square if its speed. Find expressions for the velocity v in terms of the displacement x.*

4. *A particle moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is* v*. At time* $t = 0$*,* $x = 0$ *and* $v = V$*. Subsequently the particle is slowing down at a rate is equal to* kv^3*, where* k *is a positive constant. Show that* $kx = \frac{1}{v} - \frac{1}{V}$*.*

Exercise 7.1.2

1. *A particle moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity v is given by* $\frac{1}{v} + A + Bt$. *for some positive constants A and B. Show that the retardation of the particle is proportional to the square of the speed.*

2. *A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is* v. *At time* $t = 0$ *and* $v = 1$. *Subsequently the particle experiences a retardation of magnitude* e^v . *Find the distance traveled by the particle in coming to rest.*

3. *A particle moves in a straight line with retardation which increases uniformly with the distance moved. Initially the retardation is* $5\,ms^{-2}$ *and when the particle has moved a distance of* $12\,m$ *the retardation is* $11\,ms^{-1}$. *Find the distance moved by the particle in coming to rest if the initial velocity is* $20\,ms^{-1}$.

7.1.2 Newton's Laws of Motion

Newton's Laws of Motion:

- If the resultant force on a particle is zero, the particle remains at rest, or continues to move in a straight line with constant velocity.
- If the resultant force on a particle in not zero, the observed acceleration is related to the resultant force by the vector equation $\overrightarrow{F} = m\overrightarrow{a}$. I m and a are measured in kg and ms^{-2} respectively, F is measured in newtons (N).
- When the motion of the particle is restricted to one dimension, Newton's second law can be written $F = m\ddot{x}$.

Example 7.1.2 A particle of mass m moves in a straight line subject to a resistance force of magnitude $mk(v + v^3)$, where v is the velocity of the particle. Initially the particle has velocity V. If x is the displacement of the particle from its initial position at time t, derive expressions from x in terms of v, and for v^2 in terms of t. Deduce the limiting values of v and x as $t \to \infty$

Solution: *initial position and direction of motion* $F = -mk(v + v^3) \;\Rightarrow\; \ddot{x} = -k(v + v^3)$

$t = 0 \;\Rightarrow\; x = 0$ *and* $v = V$.

Expression relating x and v:

$\ddot{x} = -k(v + v^3) \;\Rightarrow\; v\frac{dv}{dx} = -kv(1 + v^2) \;\Rightarrow\; -k\frac{dx}{dv} = \frac{1}{1+v^2}$

$-kx = \tan^{-1} v + c$, *constant.*

$x = 0,\ v = V \;\Rightarrow\; c = -\tan^{-1} V \;\Rightarrow\; kx = \tan^{-1} V - \tan^{-1} v$

$\Rightarrow\; x = \frac{1}{k}(\tan^{-1} V - \tan^{-1} v)$ ...*(1)*

Expression relating v and t:

$\ddot{x} = -k(v + v^3) \;\Rightarrow\; \frac{dv}{dt} = -kv(1 + v^2) \;\Rightarrow -k\frac{dt}{dv} = \frac{1}{v(1+v^2)}$ *(partial fractions)*

$-kt = \ln Av - \frac{1}{2}\ln(1 + v^2) \;\Rightarrow\; -2kt = \ln\left(A^2\frac{v^2}{1+v^2}\right)$, *A constant.*

$t = 0,\ v = V \;\Rightarrow\; A^2 = \frac{1+V^2}{V^2}$

$\therefore e^{-2kt} = \frac{1+V^2}{V^2}\frac{v^2}{1+v^2} \;\Rightarrow\; \frac{v^2}{1+v^2} = \frac{V^2e^{-2kr}}{1+V^2}$

$\Rightarrow\; v^2 = \frac{V^2e^{-2kt}}{1+V^2(1-e^{-2kt})}$, ...*(2)*

From (2) $v = 0$ *as* $t \to \infty$, $\therefore v \to 0^+$ *at* $t \to \infty$.

Then from (1), $x \to \left(\frac{1}{k}\tan^{-1} V\right)$ *as* $t \to \infty$.

Exercise 7.1.3

1. *A particle moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is v. At time $t = 0$ and $v = V$. Subsequently the particle is slowing down at a rate proportional to the square of its speed. Find expressions for the velocity v and the displacement x in terms of the time t.*

2. *A particle of mass m moves in a horizontal straight line. The particle is resisted by a constant force mk and a variable force mv^2, where k is a positive constant and v is the speed. When $t = 0$, $v = u$. Find the distance traveled and the time taken as the particle is brought to rest.*

Exercise 7.1.4

1. *A particle of mass m moves in a horizontal straight line. The particle is resisted by a constant force* $2m$ *and a variable force* mv *, where* v *is the speed. When* $t = 0,\ v = 4$*. Find the distance travelled and the time taken for the particle to come to rest.*

2. *A particle of mass m is set in motion with speed* u*. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude* $mk(1 + v^2)$ *where* k *is a constant and* v *is its speed at time* t*.*
   (a) *Show that the particle is brought to rest after a time* $\frac{1}{k}\tan^{-1} u$*;*

   (b) *Find an expression for the distance travelled by the particle in this time.*

7.1.3 Simple Harmonic Motion

Definition: If a particle movies is a straight subject to a force directed towards a fixed point O, with magnitude proportional to the distance of the particle from o, the ensuing motion is oscillatory with O the centre of oscillation, and is termed simple harmonic.
The force acting on the particle is described as a restoring force.
If we choose O as the origin, the restoring force is given by $F = -mn^2x$, where m is the mass of the particle and n is a positive constant. Hence , using Newton's second law, the equation of motion is $\ddot{x} = -n^2x$.
the equation has solution $x = A\cos(nt+\alpha)$, where A is called the amplitude of the oscillation, while the time taken foe one complete oscillation T is called the period, where $T = \frac{2\pi}{n}$,
Differentiation gives $v = -nA\sin(nt+\alpha)$, and hence $v^2 = n^2(A^2 - x^2)$.

Example 7.1.3 A man sat on a wharf finish form high tide at 7.00 am until low tide at the 1.10 pm. At 7.00 am he notice that the top of a buoy was 0.5 m above the level of the wharf, while at 1.10 pm it was 3.5 m below the level of the wharf. If the motion of the tide is assumed to be simple harmonic, at what time was the top of the buoy level with the wharf? what is the maximum speed of the buoy?

Solution: *Period is* $2 \times 370 = 740$ *minutes, Amplitude is* $\frac{1}{2}(0.5 + 3.5) = 2m$.
Motion is simple harmonic $\Rightarrow\ \ddot{x} = -n^2x,\ n = \frac{2\pi}{740}$
This equation has solution $x = 2\cos(nt+\alpha),\ 0 \le \alpha < 2\pi$
Initial conditions: $t = 0,\ x = 2\ \Rightarrow\ \cos\alpha = 1\ \Rightarrow\ \alpha = 0,$
$\therefore x = 2\cos(\frac{\pi}{370}t)$ *and* $v = -\frac{\pi}{185}\sin(\frac{\pi}{370}t).$
Buoy level with wharf $\Rightarrow\ x = 1.5\ \therefore \cos(\frac{\pi}{370}t) = 0.75$
The first such t is $t = \frac{370}{\pi}\cos^{-1}0.75 = 85.$
Hence the top of the buoy is level with the wharf at approximately $8.25\,am$.
$v = -\frac{\pi}{185}\sin(\frac{\pi}{370}t)\ \Rightarrow$ *maximum* v *is* $\frac{\pi}{185} \approx 0.01698\,m/min.$
Hence the maximum speed of the buoy is $2.8 \times 10^{-4}ms^{-1}$.

Exercise 7.1.5

1. *On a certain day, low water for a harbour occurs at* $.30\,m$ *and high water at* $9.45\,m$ *the corresponding depths of water being* $5\,m$ *and* $15\,m$*. Find the rate at which the level of water is rising or falling when the depth of water is* $13\,m$*.*

2. *The depth of water in a harbour is* $7.2\,m$ *at low water and* $13.6\,m$ *at high water. On Monday, low water is at* $2.05\,pm$ *and high water at* $8.20\,pm$*. The captain of a ship drawing* $12.3\,m$ *of water wants to leave harbour as early on Monday afternoon as he can. Find between what times he can leave on Monday.*

Exercise 7.1.6

1. *A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is* v. *At time* $t = 0, x = 0$ *and* $v = V$. *Subsequently the particle is slowing down at a rate equal to* kv^3, *where* k *is a positive constant. Show that* $t = \frac{x}{V} + \frac{1}{2}kx^2$.

2. *A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is* v. *At time* $t = 0,\ x = 0$ *and* $v = 1$. *Subsequently the particle experiences a retardation of magnitude* e^v . *Find the time* t_1 *for the particle to slow to half its initial and the further time* t_2 *for the particle to come to rest. Deduce that* $\frac{t_2}{t_1} = e^{\frac{1}{2}}$.

7.2 Motion of a particle moving in a straight line in a resisting medium under gravity

Upward motion: $R\downarrow mg\downarrow \Rightarrow \ F = mg + R\downarrow$

Choose projection point as origin, $\uparrow$ as positive direction $\Rightarrow \therefore \ddot{x} = -(g + \frac{R}{m})$

Downward motion: $R\uparrow \ mg\downarrow \Rightarrow \ F = mg - R\downarrow$

Choose highest point as origin, $\downarrow$ as positive direction

$\therefore F = mg - R$ (since $F\downarrow$) $\therefore \ \ddot{x} = g - \frac{R}{m}$

The terminal velocity is the limiting value of v as $\ddot{x} \to 0$

Example 7.2.1 A particle of mass is projected vertically upward in a medium where the resistance to the motion has magnitude $R = mkv$.

1. *Find the maximum height, and the time taken to reach this maximum height, in terms of the velocity of projection V.*

Solution: *Upward motion: Origin is point of projection, $\uparrow$ is positive direction,*

Initial conditions: $t = 0, \ x = 0, v = V$.

$\downarrow R = mkv, \downarrow mg \ \Rightarrow \ F = -(mg + mkv)$

Equation of motion $\ddot{x} = -(g + kv)$

Expression relating x and v: $\ddot{x} = -(g + kv) \ \Rightarrow \ v\frac{dv}{dx} = -(g + kv)$

$\frac{dx}{dv} = \frac{-v}{g+kv} \ \Rightarrow \ k\frac{dx}{dv} = -1 + \frac{g}{g+kv} \ \Rightarrow \ \frac{k^2}{g}\frac{dx}{dv} = -\frac{k}{g} + \frac{k}{g+kv}$

$\frac{k^2}{g}x = -\frac{k}{g}v + \ln(g + kv) + c$ *and* $0 = -\frac{k}{g}V + \ln(g + kV) + c$

as (c constant, initially $x = 0, \ v = V$*)* $\Rightarrow \therefore \ \frac{k^2}{g}x = \frac{k}{g}(V - v) - \ln\left\{\frac{g+kV}{g+kv}\right\} \ldots(1)$

Expression relating v and t: $\ddot{x} = -(g + kv), \ \frac{dv}{dt} = -(g + kv) \ \Rightarrow \ \frac{dt}{dv} = -\frac{1}{g+kv}$

$-k\frac{dt}{dv} = \frac{k}{g+kv} \ \Rightarrow \ -kt = \ln\{A(g + kv)\}$, *A constant,*

$t = 0, \ v = V \ \Rightarrow \ A(g + kV) = 1. \ \Rightarrow \therefore kt = \ln\left\{\frac{g+kV}{g+kv}\right\} \ldots(2)$.

At maximum height h , v = 0. Let T be the time to reach maximum height,

From (1) $\Rightarrow \ \frac{k^2}{g}h = \frac{k}{g}V - \ln(1 + \frac{k}{g}V)$ *and (2)* $\Rightarrow \ kT = \ln(1 + \frac{k}{g}V)$

$\therefore h = \frac{g}{k^2}\left\{\frac{k}{g}V - \ln(1 + \frac{k}{g}V)\right\}$ *and* $T = \frac{1}{k}\ln\left(1 + \frac{k}{g}V\right)$ (3).

Downward motion: Origin is at maximum height, $\downarrow$ positive direction,

Initial condition: t = 0, x = 0, v = 0. Terminal velocity: As $\ddot{x} \to 0, \ v \to (\frac{g}{k})^-$

Expression relating x and v: $\ddot{x} = g - kv, \ \Rightarrow \ v\frac{dv}{dx} = g - kv$

$-k\frac{dx}{dv} = \frac{-kv}{g-kv} \ \Rightarrow \ -k\frac{dx}{dv} = 1 - \frac{g}{g-kv} \ \Rightarrow \ -\frac{k^2}{g}\frac{dx}{dv} = \frac{k}{g} + \frac{-k}{g-kv}$.

$\therefore -\frac{k^2}{g}x = \frac{k}{g}v + \ln(g - kv) + c$

$x = 0, v = 0 \ \Rightarrow \ c = -\ln g, \ \therefore -\frac{k^2}{g}x = \frac{k}{g}v + \ln(1 - \frac{k}{g}v)\ldots(4)$

Expression relating v and t: $\ddot{x} = g - kv \ \Rightarrow \ \frac{dv}{dt} = g - kv$

$-k\frac{dt}{dv} = \frac{-k}{g-kv} \ \Rightarrow -kt = \ln\{A(g - kv)\}$,

A constant, $t = 0, \ v = 0 \ \Rightarrow \ Ag = 1 \ \Rightarrow \therefore kt = -\ln(1 + \frac{k}{g}v)\ldots(5)$.

2. *If the speed of the projection is twice the terminal velocity.*

   (a) *Show that the velocity of the particle on return to its projection point is given by* $\frac{kv}{g} + 2 - \ln 3 + \ln(1 - \frac{kv}{g}) = 0$. *Use Newton's method to show that an approximate solution to this equation is* $\frac{kv}{g} \approx 0.82$, *and deduce the percentage of its terminal velocity that the particle hs acquired on return to its projection point.*

Solution: *If* $V = \frac{2g}{k}$, $h = \frac{g}{k^2}(2 - \ln 3)\ldots$*from (3)*
$\therefore$ *when the particle returns to its projection point,* $x = h = \frac{g}{k^2}(2 - \ln 3)$
$\therefore \frac{k}{g}v + (2 - \ln 3) + \ln\left(1 - \frac{k}{g}v\right) = 0 \because$ *from (4)*
Let $\lambda = \frac{k}{g}v$ *and* $f(\lambda) = \lambda + \ln(1 - \lambda) + 2 - \ln 3$, *then* $f'(\lambda) = 1 - \frac{1}{1-\lambda}$.
Using the Newton's method with first approximation $\lambda_1 = 0.82$,
the second approximation is $\lambda_2 = 0.82 - \frac{f(0.82)}{f'(0.82)} = 0.82$.
Hence $\frac{kv}{g} \approx 0.82 \;\Rightarrow\; 0.82\left(\frac{g}{k}\right)$, *and the particle has acquired 82% of terminal velocity on the return to its projection point.*

   (b) *Find the ratio of the time taken to reach maximum height to the time to fall from maximum height tio the point projection.*

Solution: *Using (5), the time taken to fall from the maximum height to the projection point is given by* $kt \approx -\ln(1 - 0.82) = \ln\left(\frac{100}{18}\right)$
Using (3) , $V = \frac{2g}{k} \;\Rightarrow\; kT = \ln 3$
Hence $\frac{\text{time to maximum height}}{\text{time to fall to projectio point}} \approx \frac{\ln 3}{\ln(\frac{100}{18})} \approx 0.64$
Note that during the upward or downward motion we could find x in terms of t by integrating with respect to t the expression for v in terms of t.
For example, during the downward motion from (5)
$kt = -\ln\left(1 - \frac{k}{g}v\right) \;\Rightarrow\; \frac{k}{g}v = 1 - e^{-kt}$
$\frac{k}{g}\frac{dx}{dt} = 1 - e^{kt} \;\Rightarrow\; \frac{k}{g}x = t + \frac{1}{k}e^{-kt} + c$, c *constant.*
But $t = 0, x = 0 \;\Rightarrow\; c = -\frac{1}{k} \quad \therefore x = \frac{g}{k}t - \frac{g}{k^2}(1 - e^{-kt})$
Hence $x = \frac{g}{k^2}[kt - (1 - e^{-kt})]$.

Example 7.2.2 A particle is project vertically upward in a medium where the resistance is proportional to the square of the velocity. The velocity of projection is V, the mass of the particle is m and the initial resistance is mkV^2.

1. *Find in terms of V and k the maximum height attained and the time to reach this maximum height.*

Solution: *Upward motion: Origin at point of projection* $\uparrow$ *positive direction,*

Initial condition: $t = 0,\ x = 0,\ v = V.$

Expression relating x and v: $\ddot{x} = -(g + kv^2) \ \Rightarrow\ \frac{1}{2}\frac{dv^2}{dx} = -(g + kv^2)$

$-2k\frac{dx}{dv^2} = \frac{k}{g+kv^2} \ \Rightarrow\ -2kx = \ln[A(g + kv^2)],$ *A constant*

$x = 0,\ v = V \ \Rightarrow\ a(g + kV^2) = 1\ - 2kx = \ln\left(\frac{g+kv^2}{g+kV^2}\right)$

$\therefore\ x = \frac{1}{2k}\ln\left(\frac{g+kV^2}{g+kv^2}\right) \ldots$ *(1)*

$\downarrow R = mkv^2 \ \downarrow mg \ \Rightarrow\ F = -(mg + mkv^2)$

Equation of motion $\ddot{x} = -(g + kv^2)$

Expression relating v and t: $\dot{x} = -(g + kv^2) \ \Rightarrow\ \frac{dv}{dt} = -(g(kv^2)$

$-\frac{dt}{dv} = \frac{1}{g+kv^2} \ \Rightarrow\ \sqrt{gk}\frac{dt}{dv} = \frac{\sqrt{gk}}{g+kv^2} \ \Rightarrow\ -\sqrt{gk}t = \tan^{-1}\sqrt{\frac{k}{g}}v + c.$ *c constant.*

$t = 0,\ v = V, \Rightarrow\ V = c = -\tan^{-1}\sqrt{\frac{k}{g}}V,$

$\therefore\ \sqrt{gk}t = \tan^{-1}\sqrt{\frac{k}{g}}V - \tan^{-1}\sqrt{\frac{k}{g}}v. \ldots$ *(2)*

At maximum height, $v = 0,$ *Let the maximum height be h and the time taken to maximum height be T.*

From (1) and (2) $\Rightarrow\ h = \frac{1}{2k}\ln(1 + \frac{k}{g}V^2)$ *and* $T = \frac{1}{\sqrt{gk}}\tan^{-1}\sqrt{\frac{k}{g}}V \ldots$ *(3)*

Downward motion: Origin at maximum height $\downarrow$ *positive direction,*

Initial conditions: $t = 0, x = 0 v = 0.$

Terminal velocity: as $\ddot{x} \rightarrow 0,\ v \rightarrow \sqrt{\frac{g}{k}}.$

Expression relating x and v: $\ddot{x} = g - kv^2 \ \Rightarrow\ \frac{1}{2}\frac{dv^2}{dx} = g - kv^2$

$-2k\frac{dx}{dv^2} = \frac{-k}{g-kv^2} \ \Rightarrow\ -2kx = \ln[B(g - kv^2)],$

$x = 0,\ v = 0 \ \Rightarrow\ Bg = 1 \ \Rightarrow\ -2kv = \ln\left(1 - \frac{k}{g}v^2\right) \ldots$ *(4)*

Expression relating v and t: $\ddot{x} = g - kv^2, \ \Rightarrow\ \frac{dv}{dt} = g - kv^2$

$\therefore \sqrt{k}\frac{dt}{dv} = \frac{\sqrt{k}}{g-kv^2} = \frac{1}{2\sqrt{g}}\left\{\frac{\sqrt{k}}{\sqrt{g}-v\sqrt{k}} + \frac{\sqrt{k}}{\sqrt{g}+v\sqrt{k}}\right\}$

$2\sqrt{gk}\frac{dt}{dv} = \frac{\sqrt{k}}{\sqrt{g}-v\sqrt{k}} + \frac{\sqrt{k}}{\sqrt{g}+v\sqrt{k}} \ \Rightarrow\ 2\sqrt{gk}t = \ln\left\{\frac{\sqrt{g}+v\sqrt{k}}{\sqrt{g}-v\sqrt{k}}C\right\}$

$t = 0,\ v = 0 \ \Rightarrow\ C = 1 \ \Rightarrow \therefore 2\sqrt{gk}t = \ln\left\{\frac{\sqrt{g}+v\sqrt{k}}{\sqrt{g}-v\sqrt{k}}\right\} \ldots$ *(5)*

2. *If the speed of projection is half the terminal velocity, find :*

   (a) *the ratio of the velocity on return to the projection point to the velocity of projection.*

Solution: *If* $V = \frac{1}{2}\sqrt{\frac{g}{k}}$ *when the particle returns to its position of projection , x = h,*
$\therefore 2kx = 2kh = \ln(\frac{5}{4}) \;\Rightarrow\; \ln\left(1 - \frac{k}{g}v^2\right) \;\Rightarrow\; \ln(1 - \frac{k}{g}v^2) = -\ln(\frac{5}{4})$
From (3) and (4) $\frac{k}{g}v^2 = \frac{1}{5} \;\Rightarrow\; \frac{v}{V} = \frac{2}{\sqrt{5}}$.

   (b) *If the ratio of the time taken to fall from the maximum height to the point of projection, to the time taken to attain maximum height.*

Solution: $V = \frac{1}{2}\sqrt{fracgk}, \;\; \sqrt{gk}T = \tan^{-1}(\frac{1}{2},$ *from (3)*
and on the return to the projection point
$v = \frac{1}{\sqrt{5}}\sqrt{\frac{g}{k}} \;\Rightarrow\; 2\sqrt{gk}t = \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)$ *from (5)*
$\therefore \frac{t}{T} = \frac{1}{2}\ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)\left\{\tan^{-1}(\frac{1}{2}\right\}^{-1} \approx 1.5.$

   (c) *When and where the particle attains* 80% *of its terminal velocity.*

Solution: $v = 0.8\sqrt{\frac{g}{k}} \;\Rightarrow\; -2kx = \ln(0.36)$ *and* $2\sqrt{gk}t = \ln 9$ *from (4) and (5).*
The particle archives 80% *of its terminal velocity at a distance*
$x - h = \frac{1}{2k}\left\{-\ln(0.36) - \ln(\frac{5}{4})\right\} = \frac{1}{2k}\ln(\frac{20}{9})$ *below the point and at time*
$t + T = \frac{1}{\sqrt{gk}}\left\{\ln 3 + \tan^{-1}(\frac{1}{2})\right\}.$

Exercise 7.2.1

1. *A particle of mass m moves in a horizontal straight line. The only force acting on the particle is a resistance of magnitude* mkv^3 *where* v *is its speed and* k *is a positive constant. At time* t *the distance from a fixed point on the line is* x*. When* $t = 0$, $x = 0$ *and* $v = u$ *. Show that* $v = \frac{u}{\sqrt{(1+2ku^2t)}} = \frac{u}{1+kux}$.

2. *The force of attraction experienced by a particle of mass* m *at a distance* $(x > r)$ *from the center O of the earth towards O is* $\frac{mgr^2}{x^2}$ *, where* r *is the radius of the earth. A particle of mass* m *starts from the surface with speed* u *directly away from O.*

   (a) *Find the subsequent speed when the particle is distance x from O.*

   (b) *Deduce that the particle will escape from the attraction of the earth if* $u^2 > 2gr$.