4 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
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4 Topic 4 — Integration Part 4

4.10 Recurrence Formulae

- Repeated application of integration by parts can be used to reduce the power of a function in the integrand stepwise until the integral is know.
- Using the recurrence formula is equivalent to repeated application of integration by parts.

Example 4.10.1

1. *If* $I_n = \int x^n e^x\,dx.$, *show that* $I_n = x^n e^x - nI_{n-1}$, *and hence find* $\int x^3 e^x\,dx$.

Solution:

$$\int x^n e^x\,dx = x^n e^x - \int nx^{n-1}e^x\,dx, \qquad \textit{(integration by parts)}$$

$$I_n = x^n e^x - nI_{n-1}\ * \qquad \textit{(recurrence formula)}$$

$$\textit{Hence } I_3 = x^3e^x - 3I_2, \qquad \textit{(*with n = 3)}$$

$$= x^3e^x - 3(x^2e^x - 2I_1), \qquad \textit{(* with n = 2)}$$

$$= (x^3 - 3x^2)e^x + 6(xe^x - 1I_0) \qquad \textit{(* with n = 1)}$$

$$\therefore I_3 = (x^3 - 3x^2 + 6x)e^e - 6I_0$$

$$\textit{Then } I_0 = \int e^x\,dx = e^x + c,$$

$$\textit{Gives } I_3 = (x^3 - 3x^2 + 6x - 6)e^x + c.$$

$$\therefore \int x^3e^x\,dx = (x^3 - 3x^2 + 6x - 6)e^x + c$$

2. *If* $I_n = \int \tan^n x\,dx$ *for* $n \geq 0$, *show that* $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$ *for* $n \geq 2$.

Solution:

Let $n \geq 2$. *Using the pattern* $\int f^n(x)f'(x)\,dx = \dfrac{f^{n+1}x}{n+1} + c,$

$$\textit{We get } I_n = \int \tan^n x\,dx = \int \tan^{n-2}x\tan^2 x\,dx$$

$$= \int \tan^{n-2}x\left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \int \tan^{n-2}x\frac{1}{\cos^2 x}\,dx - \int \tan^{n-2}x\,dx$$

$$= \frac{\tan^{n-1}x}{n-1} - I_{n-2},, \quad \textit{where} \begin{cases} I_0 = \int dx = x + c, \\ I_1 = \int \tan x\,dx \\ \quad = -\ln|\cos x| + c. \end{cases}$$

Exercise 4.10.1

1. *If* $I_n = \int_1^e (\ln x)^n \, dx \ for \ n \geq 0,$ *show that* $I_n = e - nI_{n-1} \ for \ n \geq 1.$ *Hence evaluate* I_4.

2. *If* $I_n = \int x(\ln x)^n \, dx$ *for* $n \geq 0,$ *show that* $I_n = \frac{x^2}{2}(\ln x)^n - \frac{n}{2} I_{n-1}$ *for* $n \geq 1.$

Exercise 4.10.2 If $I_n = \int_0^1 x^n \sqrt{1-x}\, dx,\ \ n = 1,\ 2\, 3\, \ldots.$

1. *Show that* $I_n = \frac{2n}{2n+3} I_{n-1}$.

2. *Hence evaluate* $\int_0^1 x^3 \sqrt{1-x}\, dx$.

3. *Show that* $I_n = \frac{n!(n+1)!}{(2n+3)!} 4^{n+1}$.

Exercise 4.10.3 If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx, \;\; n = 0, \, , 1, \, 2, \, \ldots$**.**

1. *Show that* $I_n = \frac{n-1}{n} I_{n-2}$.

2. *Hence evaluate* $\int_0^{\frac{\pi}{2}} \cos^4 x\, dx$ *and* $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$.

4.11 Further Properties of Definite Integrals

- While the indefinite integral $\int f(x)\,dx$ is a function of the variable in the integrand, the definite integral $\int_a^b f(x)\,dx$ is a constant obtained by evaluating $F(a) - F(b)$, where F'(x) = f(x).
- Clearly the value of a definite integral is independent of the variable of integration. Hence $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$, whatever the values of a and b.

Example 4.11.1

1. *Use the substitution* $u = -x$ *to evaluate* $\int_{-1}^{2} \frac{x^2}{e^x+1}\,dx$

Solution:

$$\text{Let } u = -x,\ \frac{du}{dx} = -1 \Rightarrow du = -\,dx.$$

$$\text{When } x = -2,\ \Rightarrow\ u = 2,\ \text{when } x = 2\ \Rightarrow\ u = -2$$

$$\begin{aligned}
\therefore \int_{-2}^{2} \frac{x^2}{e^x+1}\,dx &= \int_{2}^{-2} \frac{u^2}{e^{-u}+1}\,(-du) \\
&= \int_{2}^{-2} \frac{u^2 e^u}{1+e^u}\,du \\
&= \int_{-2}^{2} \frac{u^2(e^u+1-1)}{e^u+1}\,du \\
&= \int_{-2}^{2} u^2\,du - \int_{-2}^{2} \frac{u^2}{e^u+1}\,du \\
\text{But } \int_{-2}^{2} \frac{x^2}{e^x+1}\,dx &= \int_{-2}^{2} \frac{u^2}{e^u+1}\,du \\
\therefore 2\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx &= \int_{-2}^{2} u^2\,du = \frac{1}{3}\,[u]_{-2}^{2} \\
\text{Hence } \int_{-2}^{2} \frac{x^2}{e^x+1}\,dx &= \frac{1}{6}.16 = \frac{8}{3}
\end{aligned}$$

2. *Show that* $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

Solution: $\text{Let } x = a - u,\ \text{then } du = -dx,\ x = 0,\ \Rightarrow u = a,\ x = a\ \Rightarrow\ u = 0,$

$$\begin{aligned}
\text{Thus } \int_0^a f(x)\,dx &= -\int_a^0 f(a-u)\,du \\
&= \int_0^a f(a-u)\,du \\
&= \int_0^a f(a-x)\,dx.
\end{aligned}$$

Exercise 4.11.1

1. *Show that* $\int_{-a}^{a} f(x)\,dx = \int_0^a \{f(x) - f(-x)\}\ dx.$

2. *Hence deduce that* $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ *, if* $f(x)$ *is even ,*
and deduce that $\int_{-a}^{a} f(x)\,dx = 0$ *, if* $f(x)$ *is odd.*

3. *Hence evaluate* $\int_{-\pi}^{\pi} \sin^5 x \cos^8 x\,dx.$

Exercise 4.11.2

1. *Show that* $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

2. *Hence evaluate* $\int_0^\pi x\sin^2 x\,dx$.

3. *Using the result* $\int_{-a}^a f(x)\,dx = \int_0^a f(x) + f(-x)\,dx$*, show that* $\int_{-1}^1 \frac{1}{1+e^{-x}}\,dx$*, and hence evaluate the given internal.*

Exercise 4.11.3

1. *Using the result* $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx$, *show that* $\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$, *and hence evaluate the given integral.*

2. *Using the result* $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx$, *show that* $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2 x}$, *and hence evaluate the given integral.*

4.12 Past Exam Questions

Exercise 4.12.1 If $I_n = \int_0^1 \frac{x^n}{\sqrt{x^3+1}}\,dx$**.**

1. *Show that* $(2n-1)I_n + 2(n-2)I_{n-3} = 2\sqrt{2}$.

2. *Hence evaluate* I_8.

Exercise 4.12.2

1. *Show that* $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$.

2. *If* $I_n = \int_0^1 \frac{t^n}{1+t^2}\, dt$, *show that* $I_n = \frac{1}{n-1} - I_{n-2},\ n \geq 2$.

3. *Hence evaluate* I_6.

Exercise 4.12.3

1. *Given that* $I_n = \int_0^{\pi} \sin^n x\, dx$, *show that* $nI_n = (n-1)I_{n-2}$.

2. *Hence, evaluate* I_8 .

3. *Use the result* $\int_0^a f(x)\, d = \int_0^a f(a-x)\, dx$ *to show that* $\int_0^{\pi} x \sin^n x\, dx = \frac{\pi}{2} I_n$.