4 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
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4 Topic 4 — Integration Part 3

4.7 Integrals of Trigonometric Functions

The standard trigonometric identities are useful for finding some trigonometric integrals:

- $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ for integral of $\cos^2\theta$.
- $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ for integral of $\sin^2\theta$.
- $\tan^2\theta = \sec^2 - 1$ and $\cot^2\theta = \csc^2\theta - 1$ for $\tan^2\theta$ and $\cot^2\theta$ respectively.

Example 4.7.1

1. *Find* $\int \tan^3 x dx$.

Solution:
$$\begin{aligned}\int \tan^3 x dx &= \int \tan x.\tan^2 x\, dx\\ &= \int \tan x(\sec^2 x - 1)\, dx\\ &= \int (\tan x \sec^2 x - \tan x)\, dx,\ \textit{patterm} \int f(x).f'(x)\, dx = \frac{1}{2}\{f(x)\}^2\\ &= \int \left(\tan x\sec^2 x + (\frac{-\sin x}{\cos x})\right) dx\\ &= \frac{1}{2}\tan^2 x + \ln|\cos x| + c\end{aligned}$$

2. *Find* $\int \cos 6x \cos 2x\, dx$.

Solution:
$$\begin{aligned}\int \cos 6x\cos 2x\, dx &= \frac{1}{2}\int(\cos 4x + \cos 8x)\, dx\\ &= \frac{1}{2}\int \cos 4x\, dx + \frac{1}{2}\int \cos 8x\, dx\\ &= \frac{1}{8}\sin 4x + \frac{1}{16}\sin 8x + c.\end{aligned}$$

3. *Evaluate* $\int_0^{\frac{\pi}{6}} \sin 3x \cos x\, dx$.

Solution:
$$\begin{aligned}\int_0^{\frac{\pi}{6}} \sin 3x\cos x\, dx &= \frac{1}{2}\int_0^{\frac{\pi}{6}}(\sin 2x + \sin 4x)\, dx\\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}}\sin 2x\, dx + \frac{1}{2}\int_0^{\frac{\pi}{6}}\sin 4x\, dx\\ &= \left[\frac{-\cos 2x}{4} - \frac{\cos 4x}{8}\right]_0^{\frac{\pi}{6}}\\ &= \left(\frac{(-\frac{1}{2})}{4} - \frac{(-\frac{1}{2})}{8}\right) - \left(\frac{-1}{4} - \frac{1}{8}\right) = -\frac{1}{6} + \frac{3}{8} = \frac{5}{6}.\end{aligned}$$

Using the following identities to convert the product of trigonometric functions:

- $2\sin\alpha\cos\beta = \sin(\alpha-\beta)+\sin(\alpha+\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta)$

Example 4.7.2 Find $\int \cos 4x\cos 2x\, dx$.

Solution: *Using* $2\cos\alpha.\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta)$.

$$\begin{aligned}\int \cos 4x\cos 2x\, dx &= \frac{1}{2}\int (\cos 2x + \cos 6x)\, dx\\ &= \frac{1}{2}\int \cos^2\ dx + \frac{1}{2}\int \cos 6x\, dx\\ &= \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + c.\end{aligned}$$

Exercise 4.7.1

1. *Find* $\int \cos 3x \sin x\, dx$.

2. *Find* $\int \sin 2x \cos 3x\, dx$.

Exercise 4.7.2

1. *Find* $\int \sin 4x \cos x \, dx$.

2. *Find* $\int \sin^5 x \cos^4 x \, dx$.

3. $\int \frac{1+\sin x}{\cos^2 x} \, dx$.

4. *Evaluate* $\int_0^{\frac{\pi}{4}} (\tan^3 x + \tan x) \, dx$.

Exercise 4.7.3 Find the followings by using the given substitution or otherwise.

1. $\int \cos^2 x \sin^5 x \, dx, \quad (u = \cos x).$

2. $\int \frac{\sin^3 x}{\cos^5 x} \, dx, \quad (u = \cos x).$

3. $\int \sqrt[3]{\sin x} \cos^3 x \, dx, \quad (u = \sin x).$

4.8 Integration By Parts

Let $F(x)$ be a primitive fuction of f(x). Using the product rule for differentiation,

$$\tfrac{d}{dx}F(x)g(x) = f(x)g(x) + F(x)g'(x).$$

Integrating both sides with respect to x,

$$F(x)g(x) = \int f(x)g(x)\,dx + \int F(x)g'(x)\,dx.$$

$$\int \text{Product} = [\text{Integrate first leave second}] - \int [\text{integrate first, derive second}]$$

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

Example 4.8.1

1. *Find* $\int x\ln x\,dx$ *bu using* $\int u\,dv = uv - \int v\,du$.

Solution: *We note that* $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

Hence, integration by parts , with $\ln x$ *as second function,*

removes $\ln x$ *from the integrad, leaving powers of* x.

$$\begin{aligned}\therefore \int x\ln x\,dx &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\frac{1}{x}\,dx\\ &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx\\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c\\ &= \frac{1}{4}x^2(2\ln x - 1) + c.\end{aligned}$$

2. *Evaluate* $\int_0^\pi e^\pi \sin x\,dx$.

Solution:

$$\begin{aligned}\int_0^\pi e^x\sin x\,dx &= [e^x\sin x]_0^\pi - \int_0^\pi e^x\cos x\,dx\\ &= 0 - \int_0^\pi e^x\cos x\,dx\\ &= -\left\{[e^x\cos x]_0^\pi - \int_0^\pi e^x(-\sin x)\,dx\right\}\\ &= e^\pi + 1 - \int_0^\pi e^x\sin x\,dx\end{aligned}$$

$$\therefore\ 2\int_0^\pi e^x\sin\,dx = e^\pi + 1.\ \Rightarrow\ \textit{Hence}\int_0^\pi e^x\sin x\,dx = \frac{1}{2}(e^\pi+1).$$

Example 4.8.2

1. *Find* $\int \sin^{-1} x\, dx$.

Solution:

$$\int 1.\sin^{-1} x\, dx = x\sin^{-1} x - \int x\frac{1}{\sqrt{1-x^2}}\, dx$$
$$= x\sin^{-1} x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x)\, dx$$
$$\therefore \text{ using the pattern } \int \{f(x)\}^{-\frac{1}{2}} f'(x)\, dx = 2\{f(x)\}^{\frac{1}{2}}$$
$$\text{Hence } \int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

2. *Evaluate* $\int_0^1 x\sin^{-1} x\, dx$.

Solution:

$$\int_0^1 x\sin^{-1} x\, dx = \left[\frac{1}{2}x^2\sin^{-1}\right]_0^1 - \int_0^1 \frac{1}{2}x^2\frac{1}{\sqrt{1-x^2}}\, dx$$
$$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx$$
$$\text{Let } u^2 = 1 - x^2, \text{ where } u > 0$$
$$2u\frac{du}{dx} = -2x, \Rightarrow u\, du = -x\, dx.$$
$$x = 0, \Rightarrow u = 1, \; x = 1, \Rightarrow u = 0.$$
$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}}x\, dx = \int_0^1 \sqrt{1-u^2}\, du$$
$$= \int_0^1 \sqrt{1-x^2}\, dx$$
$$= \frac{\pi}{4} \text{ (Area of } \frac{1}{4} \text{ of a circle of radius 1.)}$$
$$\text{Hence } \int_0^1 x\sin^{-1} x\, dx = \frac{\pi}{4} - \frac{1}{2}\times\frac{\pi}{4}$$
$$= \frac{\pi}{8}.$$

3. *Evaluate* $\int_0^2 \ln x\, dx$.

Solution:

$$\int_0^2 \ln x\, dx = \int_1^2 1.\ln x\, dx$$
$$= [x\ln x]_1^2 - \int_1^2 x\frac{1}{x}\, dx$$
$$= 2\ln 2 - \int_1^2 1\, dx$$
$$= 2\ln 2 - 1.$$

Exercise 4.8.1

1. *Find* $\int xe^x\,dx$.

2. *Find* $\int x\sin x\,dx$.

3. *Find* $\int \tan^{-1} x\,dx$.

4. *Find* $\int x^2 \ln x\,dx$.

Exercise 4.8.2

1. *Find* $\int x\cos^2 x\,dx$.

2. *Find* $\int \ln(x^2+1)\,dx$.

3. *Find* $\int x\sec x.\tan x\,dx$.

Exercise 4.8.3

1. *Find* $\int x^2 \cos x \, dx$.

2. *Find* $\int e^x \cos x \, dx$.

3. *Evaluate* $\int_0^1 x e^{2x} \, dx$.

4. *Evaluate* $\int_1^e (\ln x)^2 \, dx$.

Exercise 4.8.4

1. *Find* $\int \ln(x^2 - 1)\, dx$.

2. *Find* $\int \frac{1}{\cos^3 x}\, dx$.

4.9 Past Exam Questions

Exercise 4.9.1

1. *Evaluate* $\int_0^{\ln 2} xe^x \, dx$.

2. *Find* $\int \frac{\ln x}{x^2} \, dx$.

3. *Find* $\int xe^{-x} \, dx$.

Exercise 4.9.2

1. *If* $I_n = \int x(\ln x)^n\,dx$ *for* $n \geq 0$*, show that* $I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}$.

2. *Hence, find* $\int x(\ln x)^2\,dx$.

3. *Evaluate* $\int_0^{\pi} e^x \cos x\,dx$.

Exercise 4.9.3

1. *If* $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x \, dx$ *for* $n \geq 0$, *show that* $I_n = \frac{n-1}{n+2} I_{n-2}$ *for* $n \geq 2$.

2. *Hence or otherwise evaluate* $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x \, dx$.