4 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
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4 Topic 4 — Integration Part 2

4.4 Reduction to standard form by algebraic rearrangement of the integrand

Definition: Some integrals can be reduced to standard form by expressing the integrand as a series of terms in rational powers of x.
When the integrand is a rational function of the form $\frac{P(x)}{Q(x)}$,
where P(x) and Q(x)are polynomials, using the division transformation and decomposition into partial fractions reduces the integral to a combination of standard forms.

Example 4.4.1 Case 1 $Q(x)$ is linear and deg $P \geq$ deg Q.

1. *Find* $\int \frac{3x-2}{2x+1}\,dx$

Solution:

Since $\frac{3x-2}{2x+1} = \frac{\frac{3}{2}(2x+1)-\frac{7}{2}}{2x+1}$ *(by inspection)*

$$= \frac{3}{2} - \frac{7}{2} \times \frac{1}{2x+1},$$

$$\therefore \int \frac{3x-2}{2x+1}\,dx = \int \left(\frac{3}{2} - \frac{7}{4} \times \frac{2}{2x+1}\right) dx$$

$$= \int \frac{3}{2}\,dx - \frac{7}{4}\int \frac{2}{2x+1}\,dx$$

$$= \frac{3}{2}x - \frac{7}{4}\ln|2x+1| + c$$

2. *Find* $\int \frac{2x^2-x+1}{x-2}\,dx$.

Solution: *By formal division*

$$\begin{array}{r|l} & 2x+3 \\ \hline x-2 & 2x^2 - x + 1 \\ & -2x^2 + 4x \\ \hline & 3x + 1 \\ & -3x + 6 \\ \hline & 7 \end{array}$$

$$\therefore \int \frac{2x^2-x+1}{x-2}\,dx = \int \left(2x+3+\frac{7}{x-2}\right) dx$$

$$= \int 2x\,dx + \int 3\,dx + 7\int \frac{1}{x-2}\,dx$$

$$= x^2 + 3x + 7\ln|x-2| + c.$$

Exercise 4.4.1

1. *Find* $\int \frac{2x+3}{x^2+2x+5}\,dx$.

2. *Evaluate* $\int_2^4 \frac{(x^2-1)^2}{x}\,dx$.

3. *Find* $\int \frac{x(2x+1)}{x+1}\,dx$.

Example 4.4.2 Case 2: $Q(x)$ is a product of linear factors as degree $P <$ degree Q.

1. *Find* $\int \frac{3x-2}{(x-1)(x-2)}\,dx$.

Solution: *Since* $\dfrac{3x-2}{(x-1)(x-2)} \equiv \dfrac{a}{x-1} + \dfrac{b}{x-2}, \Rightarrow 3x-2 \equiv a(x-2)+b(x-1)$

Putting $x=2: \Rightarrow \ b=4,$ *Putting* $x=1: \Rightarrow \ a=-1,$

$$\begin{aligned}\therefore \int \frac{3x-2}{(x-1)(x-2)}\,dx &= \int \left(\frac{-1}{x-1}+\frac{4}{x-2}\right)dx \\ &= -\ln|x-1| + 4\ln|x-2| + c \\ &= \ln\left\{\frac{(x-2)^4}{|x-1|}\right\} + c.\end{aligned}$$

2. *Find* $\int \frac{6x-10}{(x+1)(x-3)}\,dx$.

Solution: *Let* $\dfrac{6x-10}{(x+1)(x-3)} \equiv \dfrac{a}{x+1} + \dfrac{b}{x-3}.$

Then we have $6x-10 \equiv a(x-3)+b(x+1).$

Put $x=-1: \Rightarrow \ a=4,$ *Put* $x=3: \Rightarrow \ b=2.$

$$\begin{aligned} Hence \int \frac{6x-10}{(x+1)(x-3)}\,dx &= \int (\frac{4}{x+1}+\frac{2}{x-3})\,dx \\ &= 4\int \frac{1}{x+1}\,dx + 2\int \frac{1}{x-3}\,dx \\ &= 4\ln|x+1| + 2\ln|x-3| + c \\ &= \ln[(x+1)^4(x-3)^2] + c\end{aligned}$$

Exercise 4.4.2 Find $\int \frac{2x^2-2x+1}{(x-2)(x^2+1)}\,dx$

Example 4.4.3 Case 3: $Q(x)$ is a product of linear factors and degree $P \geq$ degree Q. In this case it is necessary to use the division transfomation before seeking partial fractions.

1. *Evaluate* $\int_2^3 \frac{x^2+1}{x^2-x}\,dx$.

Solution: *By division,*

$$\begin{array}{r|l} & 1, \\ x^2-x & x^2 \quad +1 \\ & -x^2+x \\ \hline & x \end{array} \qquad \Rightarrow \frac{x^2+1}{x^2-x} = 1 + \frac{x+1}{x^2-x}$$

$$\textit{Let } \frac{x+1}{x(x-1)} \equiv \frac{a}{x} + \frac{b}{x-1}, \ \Rightarrow\ x+1 \equiv a(x-1)+bx$$

$$\textit{Put } x=0 : \Rightarrow\ a=-1,\ \textit{Put } x=1 : \Rightarrow\ b=2$$

$$\begin{aligned}\therefore \int_2^3 \frac{x^2+1}{x^2-x}\,dx &= \int_2^3 \left(1 - \frac{1}{x} + \frac{2}{x-1}\right) dx \\ &= [x]_2^3 - [\ln|x|]_2^3 + 2\,[\ln|x-1|]_2^3 \\ &= 1 - \ln\frac{3}{2} + 2\ln 2 \\ &= 1 + \ln\frac{8}{3}.\end{aligned}$$

2. *Find* $\int \frac{x^2+1}{(x+1)(x+2)}\,dx$.

Solution: *By division:*

$$\begin{array}{r|l} & 1 \\ x^2+3x+2 & x^2 \quad +1 \\ & -x^2-3x-2 \\ \hline & -3x-1 \end{array} \qquad \Rightarrow \frac{x^2+1}{(x+1)(x+2)} = 1 - \frac{3x+1}{(x+1)(x+2)}$$

$$\textit{Let } \frac{3x+1}{(x+1)(x+2)} \equiv \frac{a}{x+1} + \frac{b}{x+2}, \ \Rightarrow \textit{ then we get } 3x+1 \equiv a(x+2)+b(x+1).$$

$$\textit{Put } x=-1 : \Rightarrow\ a=-2.\ \textit{put } x=-2 : \Rightarrow\ b=5.$$

$$\begin{aligned}\textit{Hence} \int \frac{x^2+1}{(x+1)(x+2)}\,dx &= \int 1\,dx - \int \frac{3x+1}{(x+1)(x+2)}\,dx \\ &= \int 1\,dx - \int (-\frac{2}{x+1} + \frac{5}{x+2})\,dx \\ &= x + 2\int \frac{1}{x+1}\,dx - 5\int \frac{1}{x+2}\,dx \\ &= x + 2\ln|x+1| - 5\ln|x+2| + c \\ &= x + \ln\left(\frac{x^2}{|(x+2)^5|}\right) + c.\end{aligned}$$

Exercise 4.4.3

1. *Find* $\int \frac{x+1}{x^2+1}\,dx$.

2. *Find* $\int \frac{x+1}{x(2x+1)}\,dx$.

3. *Find* $\int \frac{x^2}{(x+1)(x+2)}\,dx$.

Example 4.4.4 Case 4: $Q(x)$ **has an irreducible quadratic factor and degree** $P <$ **degree** Q $\triangle < 0$.

1. *Evaluate* $\int_{-1}^{0} \frac{2x+1}{x^2+2x+2}\,dx$

Solution: *As* x^2+2x+2 *is irreducible* $(\triangle < 0)$,

By completing the square gives $x^2+2x+2 = (x+1)^2+1$.

Make the substitution $u = x+1,\ du = dx,\ x=-1 \ \Rightarrow\ u=0,\ x=0,\ \Rightarrow\ u=1,$

$$\begin{aligned}\therefore \int_{-1}^{0} \frac{2x+1}{x^2+2x+2}\,dx &= \int_{-1}^{0} \frac{2x+1}{(x+1)^2+1}\,dx \\ &= \int_{0}^{1} \frac{2u-1}{u^2+1}\,du \\ &= \int_{0}^{1} \left(\frac{2u}{u^2+1} - \frac{1}{u^2+1}\right) du \\ &= \left[\ln(u^2+1)\right]_0^1 - \left[\tan^{-1} u\right]_0^1 \\ &= \ln 2 - \frac{\pi}{4}.\end{aligned}$$

2. *Evaluate* $\int_{-1}^{0} \frac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx$.

Solution: *Let* $\dfrac{7x^2-5x+4}{(x-1)(x^2+1)} \equiv \dfrac{a}{x-1} + \dfrac{bx+c}{x^2+1}$, a, b, c *are constants.*

Then $7x^2-5x+4 \equiv a(x^2+1) + (bx+c)(x-1)$

Put $x=1:\ 6=2a,\ a=3,$

Equate coefficients of $x^2:\ 7=a+b,\ \Rightarrow\ b=4,$

Equate constant terms $:\ \Rightarrow\ 4=a-c,\ \Rightarrow\ c=-1.$

$$\begin{aligned}\therefore \int_{-1}^{0} \frac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx &= \int_{-1}^{0} \left(\frac{3}{x-1} + \frac{4x-1}{x^2+1}\right) dx \\ &= \int_{-1}^{0} \left(3.\frac{1}{x-1} + 2.\frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx \\ &= \left[3\ln|x-1| + 2\ln(x^2+1) - \tan^{-1} x\right]_{-1}^{0} \\ &= -5\ln 2 - \frac{\pi}{4}.\end{aligned}$$

Exercise 4.4.4

1. *Find* $\int \frac{3x-1}{(x^2+1)(x^2+2)}\, dx$

2. *Find* $\int \frac{10}{(x-1)(x^2+9)}\, dx$.

4.5 Using substitution to reduce integrals to standard form

Definition: Trigonometric substitutions are used when the integrand involves factors like:

$$\sqrt{a^2 - x^2} \text{ or } \sqrt{x^2 \pm a^2}, \text{ since } \sin^2\theta + \cos^2\theta = 1 \text{ and } \tan^2\theta + 1 = \sec^2\theta.$$

For example:

$$\begin{cases} x = a\sin\theta, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} & \Rightarrow \sqrt{a^2 - x^2} = a\cos\theta \\ x = a\tan\theta, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \Rightarrow \sqrt{a^2 + x^2} = a\sec\theta \\ x = a\sec\theta, \ -\pi < \theta \le -\frac{\pi}{2} \text{ or } 0 \le \theta < \frac{\pi}{2} & \Rightarrow \sqrt{x^2 - a^2} = a\tan\theta \end{cases}$$

Example 4.5.1

1. *Find* $\int \frac{dx}{(9+x)\sqrt{x}}$.

Solution: *The substitution* $u = \sqrt{x}$ *reduces the integrand to a rational function.*

$$\text{Let } u = \sqrt{x}, \ \Rightarrow \ \frac{du}{dx} = \frac{1}{2\sqrt{x}}, \ \Rightarrow \ du = \frac{1}{2\sqrt{x}}\,dx,$$

$$\begin{aligned} \therefore \int \frac{1}{(9+x)\sqrt{x}}\,dx &= 2\int \frac{1}{9+x} \times \frac{1}{2\sqrt{x}}\,dx \\ &= 2\int \frac{1}{9+u^2}\,du \\ &= 2\left[\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right] + c \\ &= \frac{2}{3}\tan^{-1}\frac{\sqrt{x}}{3} + c. \end{aligned}$$

2. *Find* $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$.

Solution: *Make the substitution* $u = \sqrt{x^2+1}, \ \Rightarrow \ u^2 = x^2 + 1, \ u > 0.$

$$2u.\frac{du}{dx} = 2x, \ u\,du = x\,dx, \text{ and } \sqrt{x^2+1} = |u| = u.$$

$$\begin{aligned} \therefore \int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{x^2}{\sqrt{x^2+1}} \times x\,dx \\ &= \int \frac{u^2-1}{u}.u\,du \\ &= \int (u^2-1)\,du \\ &= \frac{1}{3}u^3 - u + c \\ &= \frac{1}{3}(x^2-2)\sqrt{x^2+1} + c. \end{aligned}$$

Exercise 4.5.1

1. *Evaluate* $\int_{-1}^{0} \frac{1}{\sqrt{3+2x-x^2}}\,dx.$

2. *Find* $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx.$

Exercise 4.5.2

1. *Find* $\int \frac{x}{x^4-1}\,dx$.

2. *Find* $\int \frac{\sec^2 x}{3-\tan x}\,dx$*, by using substitution of* $u = 3 - \tan x$.

3. *Find* $\int \frac{1}{e^x+1}\,dx$.

Exercise 4.5.3

1. *Find* $\int \csc\theta\, d\theta$, *using the substitution* $(t = \tan\frac{\theta}{2})$.

2. *Evaluate* $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\, d\theta$.

Exercise 4.5.4

1. *Evaluate* $\int_2^6 x\sqrt{6-x}\,dx$*, using the substitution* $u = 6 - x$.

2. *Evaluate* $\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$*, using the substitution* $x = \tan^2\theta$.

4.6 Past Exam Questions

Exercise 4.6.1

1. *Find* $\int \frac{1}{x^2-16x+80}\,dx$.

2. *Use the substitution* $u = x - 1$*, to find* $\int \frac{x}{\sqrt{x-1}}\,dx$.

3. *Find the exact value of* $\int_0^{\ln 3} e^x \csc^2(e^x)\,dx$.

4. *Using the substitution* $u = \frac{1}{x}$*, show that* $\int_{0.5}^{1} \frac{\ln x}{1+x^2}\,dx = \int_2^1 \frac{\ln u}{1+u^2}\,du$.

Exercise 4.6.2

1. *Using the substitution of* $u = x - 2$, *to find* $\int \frac{2x}{\sqrt{4x-x^2}}\,dx$.

2. *Find* $\int \cot x \csc^2 x\,dx$.

3. *Evaluate* $\int_0^1 \frac{x}{x^4+1}\,dx$, *using the substitution* $u = x^2$.

Exercise 4.6.3 Evaluate the following:

1. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\, d\theta$.

2. $\int_0^{2} \frac{8}{(x+2)(x^2+4)}\, dx$.

3. *Evaluate* $\int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{1}{\sqrt{(x-5)(7-x)}}\, dx$ *by using the substitution* $u = x - 6$.