4 Unit Math Homework for Year 12

Student Name: _____	**Grade**: _____
Date: _____	**Score**: _____

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This edition was printed on June 16, 2017 with worked solutions.

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4 Topic 4 — Integration Part 1

4.1 Using a Table of Standard Integrals

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	$\int \{f(x)\}^n f'(x)\,dx = \frac{1}{n+1}\{f(x)\}^{n+1} + c,\ n \neq -1$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$	$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$
$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c,\ a \neq 0$	$\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$
$\int \cos ax\,dx = \frac{1}{a}\sin ax + c,\ a \neq 0$	$\int \cos\{f(x)\} f'(x)\,dx = \sin\{f(x)\} + c$
$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c,\ a \neq 0$	$\int \sin\{f(x)\} f'(x)\,dx = -\cos\{f(x)\} + c$
$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c,\ a \neq 0$	$\int \sec^2\{f(x)\} f'(x)\,dx = \tan\{f(x)\} + c$
$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c,\ a \neq 0$	$\int \sec\{f(x)\}\tan\{f(x)\} f'(x)\,dx = \sec\{f(x)\} + c$
$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c,\ a > 0$	$\int \frac{f'(x)}{a^2+\{f(x)\}^2}\,dx = \frac{1}{a}\tan^{-1}\{\frac{1}{a}f(x)\} + c,\ a > 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c,\ a > 0$	$\int \frac{f'(x)}{\sqrt{a^2-\{f(x)\}^2}} = \sin^{-1}\{\frac{1}{a}f(x)\} + c,\ a > 0$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\lvert x + \sqrt{x^2-a^2}\rvert + c$	$\int \frac{f'(x)}{\sqrt{\{f(x)\}^2-a^2}}\,dx = \ln\lvert f(x) + \sqrt{\{f(x)\}^2-a^2}\rvert + c$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\{x + \sqrt{x^2+a^2}\} + c$	$\int \frac{f'(x)}{\sqrt{\{f(x)\}^2+a^2}}\,dx = \ln\{f(x) + \sqrt{[f(x)]^2+a^2}\} + c$

Example 4.1.1 Find the following integrals:

1. $\int \frac{1}{4+x^2}\,dx$

Solution: *Using the pattern* $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$, *with* $a = 2$,

We obtain $\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + c.$

2. $\int \frac{1}{x^2+2x+2}\,dx.$

Solution: *Make the substitution* $u = x+1,\ \Rightarrow\ du = dx,$

We obtain
$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$$
$$= \int \frac{du}{u^2+1}$$
$$= \tan^{-1} u + c = \tan^{-1}(x+1) + c.$$

3. $\int \frac{1}{\sqrt{2x-x^2}}\,dx.$

Solution: *Make the substitution* $u = x-1,\ \Rightarrow\ du = dx,$

We obtain
$$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$
$$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = \sin^{-1}(x-1) + c.$$

Exercise 4.1.1 Use the table of standard integrals to find the values of the following integrals:

1. $\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx$.

2. $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x\,dx$.

3. $\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$.

Pattern 1:

$$\frac{d}{dx}\{\ln f(x)\} = \frac{f'(x)}{f(x)} \text{ then } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Example 4.1.2 Find $\int \frac{\cos}{2+\sin x}\,dx$.

Solution:

If $f(x) = 2 + \sin x$, *then* $f'(x) = \cos x$.

and the given integral follows the pattern $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$.

$$\text{hence } \int \frac{\cos x}{2+\sin x}\,dx = \ln|2+\sin x| + c$$

$$= \ln(2+\sin x) + c.$$

Exercise 4.1.2 Find the following integrals:

1. $\int \frac{x}{1+x^2}\,dx$

2. $\int \frac{2x-1}{x^2-x-2}$

3. $\int \cot x\,dx$

Exercise 4.1.3 Find the following integrals:

1. $\int \frac{e^x}{1+e^x}\,dx$

2. $\int \frac{\sin 2x}{2+\sin^2 x}\,dx$

3. *Evaluate* $\int_{-1}^{3} \frac{1}{x^2-2x+5}\,dx$.

Pattern 2:

$$\int f(x)f'(x)\,dx = \frac{1}{2}\{f(x)\}^2 + c$$

Pattern 3:

$$\int f(x)^n f'(x)\,dx = \frac{1}{n+1}\{f(x)\}^{n+1} + c$$

Example 4.1.3 Find the following integrals:

1. *Find* $\int \frac{\ln x}{x}\,dx$

Solution: *If* $f(x) = \ln x$, *then* $f'(x) = \dfrac{1}{x}$

and the given integral follows the pattern $\displaystyle\int f(x)f'(x)\,dx = \frac{1}{2}\{f(x)\}^2 + c$

Hence $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + c.$

2. $\int \tan 2x \sec 2x\,dx.$

Solution:

$$\begin{aligned}
\textit{Using the pattern } \int f(x)^n f'(x)\,dx &= \frac{1}{n+1}\{f(x)\}^{n+1} + c \\
\textit{with } f(x) = \cos 2x,\ f'(x) &= -2\sin 2x \textit{ and } n = -2, \\
\textit{Noting that } \tan 2x &= \frac{\sin 2x}{\cos 2x} \textit{ and } \sec 2x = \frac{1}{\cos 2x}. \\
\textit{Thus we have } \int \tan 2x \sec 2x\,dx &= -\frac{1}{2}\int \frac{-2\sin 2x}{\cos^2 2x}\,dx \\
&= \frac{1}{2}(\cos 2x)^{(-2+1)} + c \\
&= \frac{1}{2\cos 2x} + c.
\end{aligned}$$

Exercise 4.1.4 Find $\int \frac{x}{(1+x^2)^2}\,dx$**.**

Exercise 4.1.5 Find the following integrals:

1. $\int x\sqrt{(1+x^2)}\,dx$

2. $\int \tan^3 x \sec^2 x\,dx$

3. $\int \sec^3 x \tan x\,dx$.

Pattern 4:

$$\int e^{f(x)} f'(x)\, dx = e^{f(x)} + c$$

Example 4.1.4 Find $\int e^{\sin x} \cos x\, dx$.

Solution:

$$f(x) = \sin x, \ \Rightarrow \ f'(x) = \cos x,$$

So it follows the pattern of $\int e^{f(x)} f'(x)\, dx = e^{f(x)} + c.$

Hence we have $\int e^{\sin x} \cos x\, dx = e^{\sin x} + c.$

Exercise 4.1.6 Find the following integrals:

1. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx.$

2. $\int e^{2x-1}\, dx.$

3. $\int 2xe^{x^2}\, dx.$

Pattern 5:

$$\int sec^2 f(x) f'(x)\, dx = tan f(x) + c$$

Example 4.1.5 Find $\int x \sec^2(x^2)\, dx$**.**

Solution:

$$\textit{With } f(x) = x^2, \Rightarrow f'(x) = 2x,$$

$$\textit{Hnece } \int x \sec^2(x^2)\, dx = \frac{1}{2}\int \sec^2(x^2).2x\, dx$$

$$= \frac{1}{2}\tan(x^2) + c.$$

Pattern 6:

$$\int \sin\{f(x)\} f'(x)\, dx = -\cos f(x) + c$$

Example 4.1.6 Find $\int e^x \sin(e^x)\, dx$**.**

Solution:

$$\textit{The given integral follows the pattern } \int \sin\{f(x)\} f'(x)\, dx = -\cos f(x) + c$$

$$\textit{With } f(x) = e^x \Rightarrow f'(x) = e^2,$$

$$\textit{Hence } \int e^x \sin(e^x)\, dx = -\cos e^x + c.$$

Pattern 7:

$$\int \cos\{f(x)\} f'(x)\, dx = \sin f(x) + c$$

Example 4.1.7 Find $\int \cos(\ln x)\frac{1}{x}\, dx$**.**

Solution:

$$\textit{Using the pattern } \int \cos\{f(x)\} f'(x)\, dx = \sin f(x) + c$$

$$\textit{With } f(x) = \ln x, \Rightarrow f'(x) = \frac{1}{x},$$

$$\textit{Hence } \int \cos(\ln x)\frac{1}{x}\, dx = \sin(\ln x) + c.$$

4.2 Miscellaneous Exercises

Exercise 4.2.1 Find the following integrals:

1. $\int (2x+3)^4\,dx$

2. $\int \frac{dx}{(x+2)^3}\,dx$.

3. $\int \frac{3x}{x^2-5}\,dx$.

4. *Evaluate* $\int_1^2 xe^{x^2}\,dx$.

Exercise 4.2.2 Find the following integrals:

1. $\int x^2 e^{x^3}\, dx$.

2. $\int e^{5x+1}\, dx$.

3. $\int e^{2x}\, dx$.

4. *Evaluate* $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{(2-\cos^2 x)}}\, dx$.

Exercise 4.2.3

1. *Evaluate* $\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}\, dx$.

2. *Find* $\int \frac{x}{x^4-1}\, dx$.

3. *Find* $\int \frac{1}{x}\sec^2(\ln x)\, dx$.

4.3 Past Exam Questions

Exercise 4.3.1 Find the following integrals:

1. *Evaluate* $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$. *[3 marks]*

2. *Evaluate* $\int_{\sqrt{2}}^3 \frac{1}{\sqrt{(x^2-1)}}\,dx$. *[3 marks]*

3. *Evaluate* $\int_0^{\ln 3} \frac{e^x}{1+e^x}\,dx$. *[3 marks]*

Exercise 4.3.2

1. *By expanding the left-hand side, show that* $\sin(5x + 4x) + \sin(5x - 4x) = 2\sin 5x \cos 4x$.

2. *Hence find* $\int \sin 5x \cos 4x \, dx$.. *[2 marks]*

Exercise 4.3.3

1. *Evaluate* $\int_0^{\frac{\pi}{4}} \cos x \sin^2 x \, dx$. *[2 marks]*

2. *Find* $\int \frac{1}{x^2+36} \, dx$. *[1 mark]*

Exercise 4.3.4

1. *Differentiate* $e^{3x}(\cos x - 3\sin x)$. *[2 marks]*

2. *Hence, or otherwise, find* $\int e^{3x}\sin x\,dx$. *[1 marks]*

Exercise 4.3.5

1. *Find* $\int \frac{e^x}{1+e^x}\,dx$. *[2 marks]*

2. *Evaluate* $\int_0^{\pi} \cos^2 3x\,dx$. *[2 marks]*