4 Unit Math Homework for Year 12

Student Name: ______	Grade: ______
Date: ______	Score: ______

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8 Topic 8 — Harder 3 Unit Topics Part 3

8.4 Inequalities between real numbers

Useful inequalities: $a^2 + b^2 = (a-b)^2 + 2ab \ \Rightarrow \ a^2 + b^2 \geq 2ab$
$(a+b)^2 = (a-b)^2 + 4ab \ \Rightarrow \ (a+b)^2 \geq 4ab$
In each case, equality holds if and only if (abbreviation iff) $a = b$

Example 8.4.1 Show that if $a,\ b,\ c$ and d are positive, then

1. $a^4 + b^4 + c^4 + d^4 \geq 4abcd$.

Solution: *Use the* $a^2 + b^2 \geq 2ab \ \Rightarrow \ a^4 + b^4 \geq 2a^2b^2$ (*equlity iff* $a^2 = b^2$)
Similarly $c^4 + d^4 \geq 2c^2d^2$ (*equlity iff* $c^2 = d^2$)
By addition $a^4 + b^4 + c^4 + d^4 \geq 2(a^2b^2 + c^2d^2)$
for $ab,\ cd,\ \Rightarrow \ (ab)^2 + (cd)^2 \geq 2(ab)(bd)$ (*equlity iff* $ab = cd$)
$\therefore a^4 + b^4 + c^4 + d^4 \geq 4abcd$ (*equlity iff* $a = b = c = d$).

2. $a^2 + b^2 + c^2 \geq ab + bc + ca$. *Deduce that if* $a + b + c = 1$, *then* $ab + bc + ca \leq \frac{1}{3}$. *State the condition for equality to hold.*

Solution: *We cannot pair off the terms as we did in part (1),*
but the symmetry of the result suggests that consider separately:
$a^2 + b^2, b^2 + c^2,\ c^2 + a^2$ *and then add the result.*
Using $a^2 + b^2 \geq 2ab,$ *in each case:*
$(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) \geq 2ab + 2bc + 2ca$
Hence $a^2 + b^2 + c^2 \geq ab + bc + ca$ (*equlity iff* $a = b = c$).
Then $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca)$
(*equlity iff* $a = b = c$).
$$\therefore a + b + c = 1 \ \Rightarrow \ ab + bc + ca \leq \frac{1}{3} \ (\textit{equlity iff } a = b = c = \frac{1}{3}).$$

Example 8.4.2

1. *Show that if* $a > 0,\ a + \frac{1}{a} \geq 2$.

Solution: *Using* $(a+b)^2 = (a-b)^2 + 4ab \ \Rightarrow\ (a+b)^2 \geq 4ab:$

$$\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2 + 4a \times \frac{1}{a} \ \Rightarrow\ \left(a+\frac{1}{a}\right)^2 \geq 4.\ (\textit{equlity iff } a = \frac{1}{a}).$$

$$\therefore a > 0 \ \Rightarrow\ (a + \frac{1}{a} \geq 2.\ (\textit{equlity iff } a = 1).$$

2. *Deduce that if* $a,\ b$ *and* c *are positive, then* $(a+b)(\frac{1}{a}+\frac{1}{b}) \geq 4$ *and* $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 9$.

Solution:

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 1+1+\left(\frac{a}{b}+\frac{b}{a}\right) \geq 2+2.\ (\textit{using part (1)},\ a \to \frac{a}{b}).$$

$$\therefore (a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4\ \ (\textit{equlity iff } \frac{a}{b} = 1,\ i.e.\ a = b)$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 1+1+1+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$$

$$\geq 3+2+2+2.$$

$$\therefore (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9,\ (\textit{equlity iff } a = b = c).$$

3. *Hence show that* $\frac{9}{a+b+c} \leq \frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. *and state the condition under which equality holds.*

Solution: *For symmetrical paris a and b, b and c, c and a and then adding:*

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq \frac{4}{a+b} \leq \frac{1}{a}+\frac{1}{b}\ (\textit{equlity iff } a = b).$$

$$\textit{similarly: } (b+c)\left(\frac{1}{b}+\frac{1}{c}\right) \geq \frac{4}{b+c} \leq \frac{1}{b}+\frac{1}{c}\ (\textit{equlity iff } b = b).$$

$$\textit{and } (c+a)\left(\frac{1}{c}+\frac{1}{a}\right) \geq \frac{4}{c+a} \leq \frac{1}{c}+\frac{1}{a}\ (\textit{equlity iff } c = a).$$

$$\textit{by addition: } \frac{4}{a+b}+\frac{4}{b+c}+\frac{4}{c+a} \leq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

$$\therefore \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ (\textit{equlity iff } a = b = c).$$

substitution: $a \to a+b,\ b \to b+c,\ c \to c+a,$ *we obtain:*

$$[(a+b)+(b+c)+(c+a)]\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \leq 9$$

$$\therefore \frac{9}{a+b+c} \leq \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Exercise 8.4.1

1. *If* $0 < a < b$ *show that* $a < \frac{a+b}{2} < b$.

2. *If* $0 < a < b$ *show that* $a < \sqrt{ab} < b$.

3. *If* $a > 0,\ b > 0$ *show that* $\frac{a+b}{2} > \sqrt{ab}$.

4. *If* $x > 0$ *show that* $x + \frac{1}{x} \geq 2$.

5. *If* $a > 0,\ b > 0$ *show that* $4ab \leq (a+b)^2$.

Exercise 8.4.2

1. *If* $a > 0,\ b > 0$ *and* $c > 0$ *show that* $(a+b)(b+c)(c+a) > 4abc.$

2. *If* $a > 0,\ b > 0,\ c > 0,$ *and* $a+b+c = 1$ *show that* $(1-a)(1-b)(1-c) \geq 8abc.$

3. *If* $a > 0,\ b > 0$ *and* $a+b = 1,$ *show that* $\frac{1}{a} + \frac{1}{b} \geq 4..$ *Hint: show that* $a+b \geq 2\sqrt{ab}.$

4. *If* $a > 0,\ b > 0$ *and* $a+b = 1,$ *show that* $\frac{1}{a^2} + \frac{1}{b^2} \geq 8.$ *Hint: show that* $a+b \geq 2\sqrt{ab}$

Exercise 8.4.3

1. *Show that* $ab + bc + ca \le a^2 + b^2 + c^2$.

2. *If* $a > 0,\ b > 0,$ *show that* $a^3 + b^3 \ge a^2b + ab^2$.

3. *Show that* $(a + b)^2 \le 2(a^2 + b^2)$.

4. *Show that* $(a^3 + b^3)^2 \le (a^2 + b^2)(a^4 + b^4)$.

Example 8.4.3 Show that $(b+c-a)(c+a-b) \leq c^2$**. Hence show that if** $a,\ b,$ **and** c **are the sides of a triangle , then** $(b+c-a)(c+a-b)(a+b-c) \leq abc.$ **What is the nature of the triangle if equality holds?**

Solution: $c^2 - (b+c-a)(c+a-b) = a^2 + b^2 - 2ab = (a-b)^2 \geq 0$

$\therefore (b+c-a)(c+a-b) \leq c^2$ *(with equality iff* $a = b$*)*

Clear the role of c in this inequality is different from that of a and b.

However the final inequality is symmetrical in a, b, and c.

This suggests combination of this inequality with two similar inequalities:

$$a \to b,\ b \to c,\ c \to a \begin{cases} (b+c-a)(c+a-b) \leq c^2 & (\text{with equality iff } a = b) \\ (c+a-b)(a+b-c) \leq a^2 & (\text{with equality iff } b = c) \\ (a+b-c)(b+c-a) \leq b^2 & (\text{with equality iff } c = a) \end{cases}$$

Since $a,\ b$ *and* c *are the side of a triangle, all the expressions involved are poistive.*

Hence mulipplication gives: $(b+c-a)^2(c+a-b)^2(a+b-c)^2 \leq c^2a^2b^2$

$\therefore\ (b+c-a)(c+a-b)(a+b-c) \leq abc,$ *(with equality iff* $a = b = c$*).*

and the triangle is equalateral.

Exercise 8.4.4

1. *Show that* $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$.

2. *Show that* $(a^2-b^2)(c^2-d^2) \leq (ac-bd)^2$. *Hence deduce that* $(a^2-b^2)(a^4-b^4) \leq (a^3-b^3)^2$.

8.5 The triangle inequalities

Definition: Inequalities involving absolute values are often related to the triangle inequalities, such as:

$$|a + b| \leq |a| + |b| \text{ and } |a - b| \geq ||a| - |b||$$

.

Example 8.5.1

1. *Show that* $|a + b + c| \leq |a| + |b| + |c|$.

   Solution: $|a + b + c| = |(a + b) + c| \leq |a + b| + |c| \leq |a| + |b| + |c|$.

2. *Show that* $|a - b| \geq ||a - c| + |b - c||$.

   Solution: $|a - b| = |(a - c) - (b - c)| \geq ||a - c| - |b - c||$.

Exercise 8.5.1 Show that $|\sin(\alpha + \beta)| \leq |\sin \alpha| + |\sin \beta|$**.**

8.6 Application of calculus to inequalities

Example 8.6.1

1. *Show that* $x \geq \ln(x+1)$ *for all* $x > -1$. *State when equality holds.*

> ***Solution:*** *Let* $f(x) = x - \ln(x+1),\ f'(x) = 1 - \dfrac{1}{x+1}$
>
> *but* $f'(x) = 0 \ \Leftrightarrow \ x = 0.$
>
> $\therefore f(x)$ *has an absolute minimum of* 0 *when* $x = 0.$
>
> *Hence for* $x > -1,\ x \geq \ln(x+1)$, (*with equality iff* $x = 0$.)

2. *Show that* $\cos x \geq 1 - \frac{x^2}{2}$ *for all* x.

> ***Solution:*** *Let* $f(x) = \cos - \left(1 - \dfrac{x^2}{2}\right)$. *(Then* $f(x)$ *is an even function.)*
>
> $f'(x) = x - \sin x \ \Rightarrow \ f''(x) = 1 - \cos x \geq 0$ *for all* x.
>
> $f(0) = 0$ *and* $f'(x)$ *is a non-decreasing function* $f'(x) \geq 0$ *for all* $x \geq 0$.
>
> $\therefore$ $f(x)$ *is a non-decreasing function for* $x > 0$,
>
> $\therefore$ $f(0) = 0$ *and* $f(x)$ *is even function* $\Rightarrow$ $f(x) \geq 0$ *for all* x,
>
> $\therefore$ $\cos x \geq 1 - \dfrac{x^2}{2}$ *for all* x.

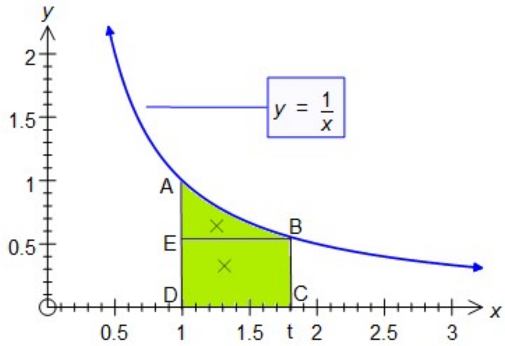
Exercise 8.6.1

1. *Show that* $e^x > x + 1$ *for all* $x > 0$.

2. *If* $a > 1$ *and* $x > -1$, *show that* $(1+x)^a \geq 1 + ax$.

Example 8.6.2 Use $\ln t = \int_1^t \frac{1}{x}\, dx$ **for** $t > 1$ **to deduce that** $1 - \frac{1}{t} \le \ln t \le \frac{1}{2}(t - \frac{1}{t})$ **for** $t \ge 1$**.**

Solution:



Area EBCD < Shaded area < area ABCD

$$\frac{1}{t}(t-1) < \int_1^t \frac{1}{x}\, dx < \frac{1}{2}\left(1 + \frac{1}{t}\right)(t-1)$$

$$1 - \frac{1}{t} < \ln t < \frac{1}{2}\left(t - \frac{1}{t}\right), \ t > 1$$

But equation hold when $t = 1$.

$$\therefore 1 - \frac{1}{t} \le \ln t \le \frac{1}{2}\left(t - \frac{1}{t}\right), \ t \ge 1.$$

Exercise 8.6.2 If $0 < t < 1$ **show that** $\frac{1}{2} < \frac{1}{1+t} < 1$**. By integrating between 0 and u, duduce that for** $0 < u < 1, \ \frac{u}{2} < \ln(1+u) < u$**.**

Exercise 8.6.3

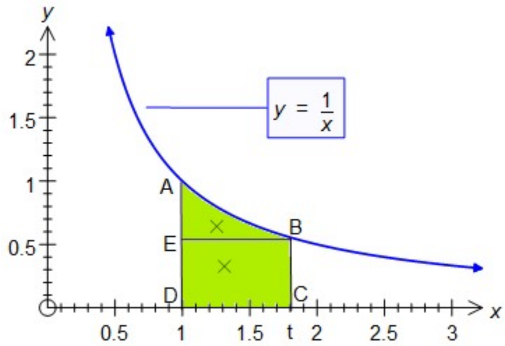
1. *Show that* $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$.

2. *If* $a > b$ *and* $b > 0$, *show that* $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{(a+b)^2}$.

3. *If* $a > 0$, *and* $b > 0$, *show that* $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$.

8.7 Past Exam Questions

Exercise 8.7.1 Let n be a positive integer greater than 1. the area of the region under the curve $y = \frac{1}{x}$ **from** $x = n - 1,\ to\ x = n$ **is between the area of two rectangles, as shown in the diagram. show that** $e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$



Exercise 8.7.2 Suppose $0 \leq t \leq \frac{1}{\sqrt{2}}$**.**

1. *Show that* $0 \leq \frac{2t^2}{1-t^2} \leq 4t^2$.

2. *Hence show that* $0 \leq \frac{1}{1-t} + \frac{1}{1+t} - 2 \leq 4t^2$.

3. *By integrating the expressions in the inequality in part (2) with respect to t from* $t = 0$ *to* $t = x$ *(where* $0 \leq x \leq \frac{1}{\sqrt{2}}$*), show that* $0 \leq \ln(\frac{1+x}{1-x}) - 2x \leq \frac{4x^3}{3}$.

4. *Hence show that for* $0 \leq x \leq \frac{1}{\sqrt{2}},\ 1 \leq \left(\frac{1+x}{1-x}\right) e^{-2x} \leq e^{\frac{4x^3}{3}}$.

Exercise 8.7.3

1. *If* $x < y < 0$*, prove that* $(x^2 + y^x(x - y) \geq (x^2 - y^2(x + y)$.

2. *Prove theat* $a^2 + b^2 \geq ab + a + b - 1$.

3. *Given that* $a \in \mathbb{R}$ *and* $a \geq -1$*, prove that* $\frac{1}{1+a} \geq 1 - a$.

4. *Prove that* $x^3 > x^2 - x + 1$ *when x>1.*

Exercise 8.7.4

1. *Differentiate* $y = \ln(1 + x)$*, and hence draw* $y = x$ *and* $y = \ln(1 + x)$ *on one set of axis, and hence using this graph, explain why* $\ln(1 + x) < x,\ forall x > 0$.

2. *Defferentiate* $y\frac{x}{1+x}$*, and hence draw* $y = \frac{x}{1+x}$ *and* $y = \ln(1 + x)$ *on one set of axis, and hence using the graph to explain why* $\frac{x}{1+x} < \ln(1 + x)$*, for all* $x > 0$.

3. *Use the inequalities of part (1) and part (2) to show that* $\frac{\pi}{8} - \frac{1}{4}\ln 2 < \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx < \frac{1}{2}\ln 2$.