4 Unit Math Homework for Year 12

Student Name: ______________________	**Grade**: ______________________
Date: ______________________	**Score**: ______________________

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6 Topic 6 — Graphs Part 3

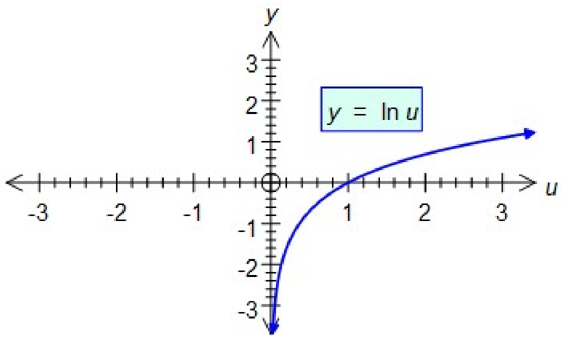
6.8 Graphs of Composite Functions

Knowledge of basic curve can often be applied to the construction of graphs of composite functions.

Example 6.8.1 Sketch the graphs $y = \ln u$ **and** $u = \cos x$, $0 \le x \le 2\pi$**. Hence sketch the graph** $y = \ln(\cos x)$, $0 \le x \le 2\pi$**.**

1. *Graph of* $y = \ln u$

Solution:

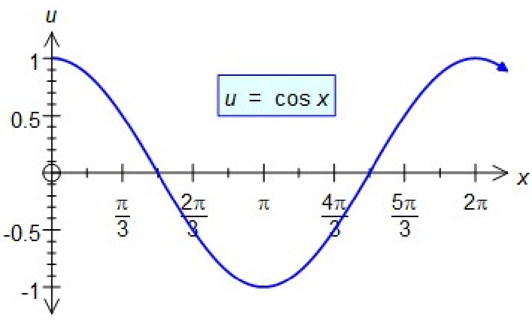


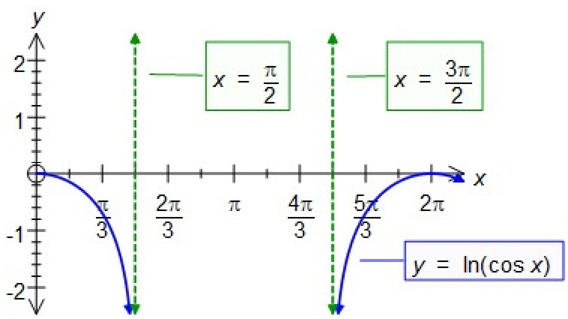
Features:

- *Vertical asymptote at u = 0*
- *y = 0 when u = 1.*
- *Monotonic increasing throughout domain {u: u>0}*

2. *Graphs of* $u = \cos x$ *and* $y = \ln(\cos x)$

Sol





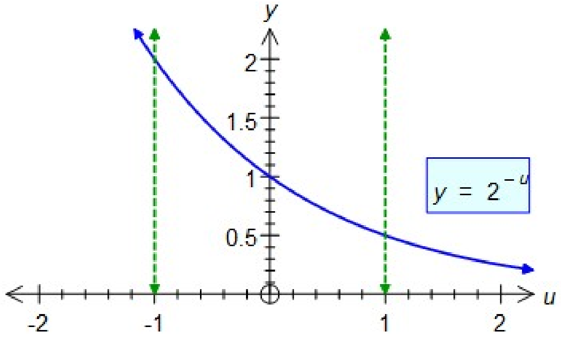
Features:

- *Because* $y = \ln u$ *is an increase function,* $y = \ln(\cos x)$ *increase as* $\cos x$ *increase and decrease as* $\cos x$ *decrease.*
- *Value of* x *for which* $\cos x \le 0$ *are not in the domain of* $y = \ln(\cos x)$.

Example 6.8.2 Sketch the graphs $y = 2^{-u}$ and $u = \sin x$, $0 \le x \le 2\pi$. Hence sketch the graph $y = 2^{-\sin x}$. $0 \le x \le 2\pi$, showing the coordinates of any stationary points.

1. *Graph of $y = 2^{-u}$.*

Solution:

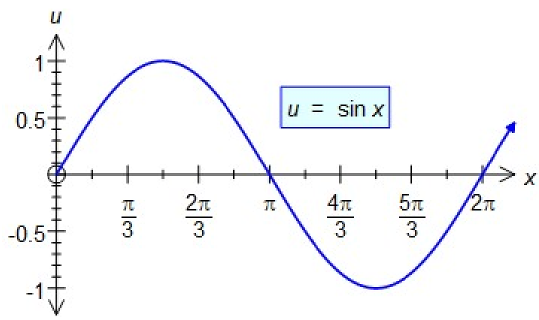


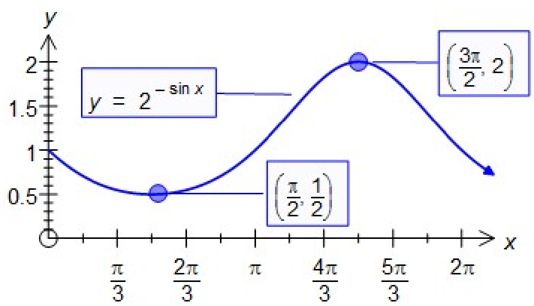
Features of $y = 2^{-u}$:

- *When $u = 0,\ y = 1$*
- *Monotonic decreasing*
- $-1 \le u \le 1,\ \Rightarrow\ 2 \ge y \ge \frac{1}{2}$

2. *Graphs of $u = \sin x$ and $y = 2^{-\sin x}$.*

Solution:



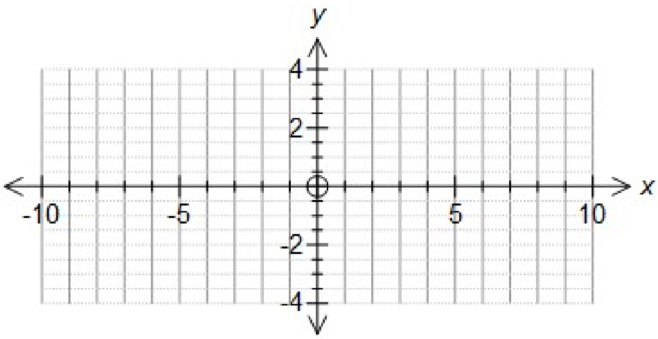


Features of $y = 2^{-\sin x}$:

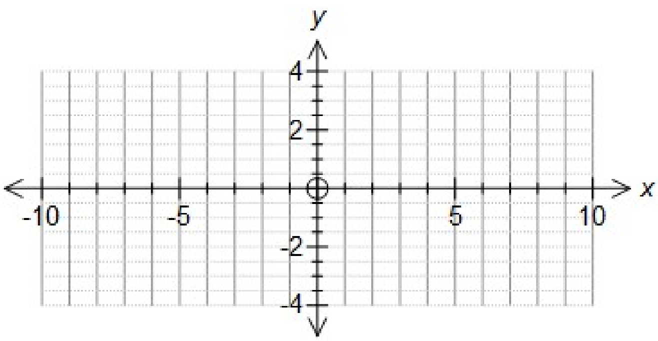
- *$y = 2^{-u}$ is a decreasing function and hence $y = 2^{-\sin x}$ decrease as $\sin x$ increase, and coversely.*
- *Thus maximum turning points of $u = \sin x$ coorespond to minimum turning points of $y = 2^{-\sin x}$ and vice versa.*

Exercise 6.8.1

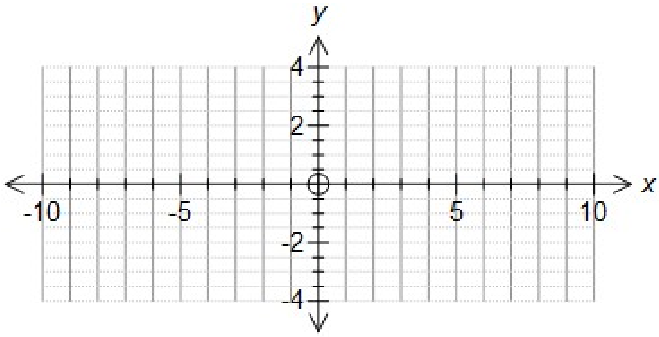
1. *Use the graphs of* $y = 2^{-u}$ *and* $u = |x|$ *(an even function) to sketch the graph of* $y = 2^{-|x|}$



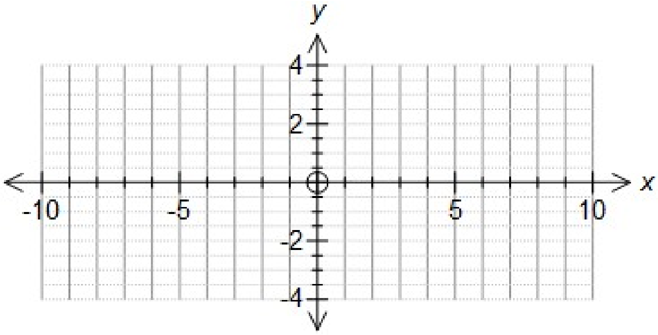
2. *Sketch the graph of* $y = e^{-x^2}$



Exercise 6.8.2 Use the graphs of $y = \ln u$ and $u = 1 - x^2$ (an even function) to sketch the graph of $y = \ln(1 - x^2)$.

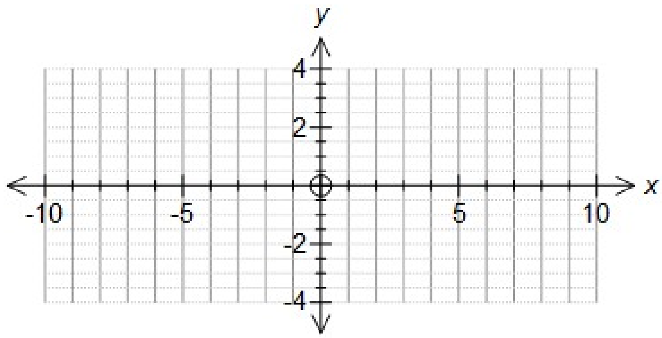


Exercise 6.8.3 Use the graphs of $y = \ln u$ **and** $u = \sin x$ $(0 \leq x \leq 2\pi)$ **to sketch the graph of** $y = \ln(\sin x)$ $(0 \leq x \leq 2\pi)$ **.**

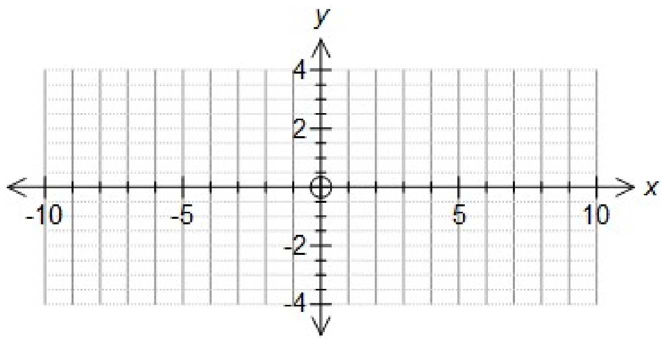


Exercise 6.8.4

1. *Use the graphs of* $y = 2^u$ *and* $u = \cos x\,(0 \le x \le 2\pi)$ *to sketch the graph of* $y = 2^{\cos x}$ $(0 \le x \le 2\pi)$



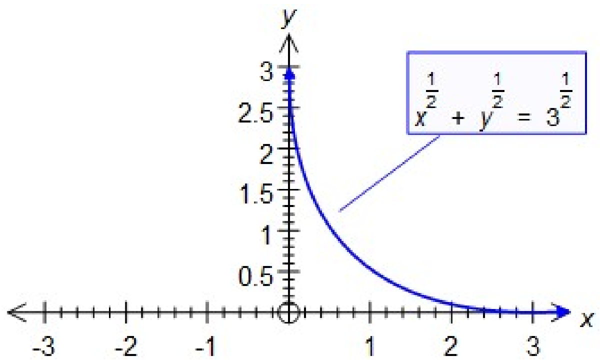
2. *Use the graphs of* $y = \ln u$ *and* $u = x^2 - 3$ *to sketch the graph of* $y = \ln(x^2 - 3)$.



6.9 Implicit Differentiation and Curve Sketching

Example 6.9.1 Sketch the graph $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3^{\frac{1}{2}}$**.**

Solution:



Take the derivative of both sides with respect to x, remembering that y is a function of x, and using the chain rule.

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3^{\frac{1}{2}} \;\Rightarrow\; \tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{2}y^{-\frac{1}{2}}\tfrac{dy}{dx} = 0$$

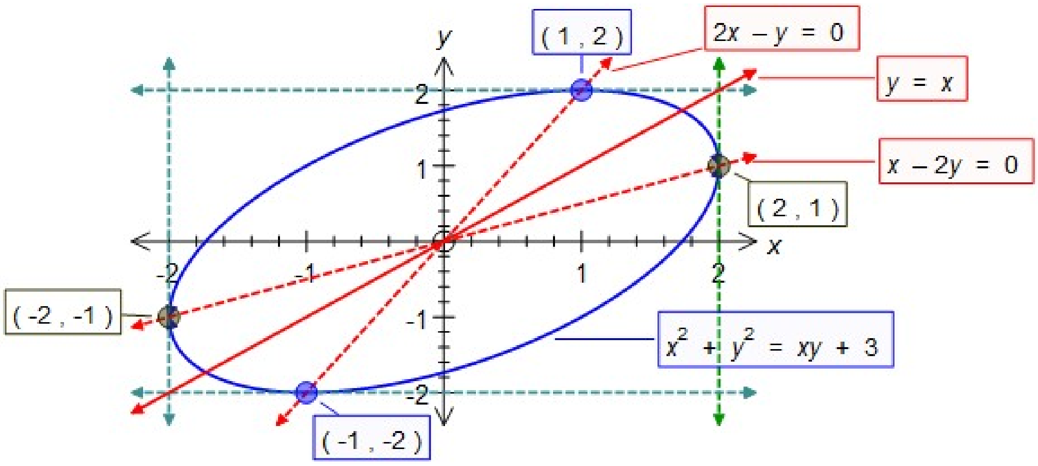
$$\therefore \tfrac{dy}{dx} = (\tfrac{y}{x})^{\frac{1}{2}}.$$

Features:

- *The function has domain* $\{x : 0 \le x \le 3\}$.
- *As* $x \to 0^+,\ y \to 3^-$, *and* $\frac{dy}{dx} \to -\infty$.
- *The curve has a vertical tangent line at the critical point* $(0, 3)$.
- *As* $x \to 3^-,\ y \to 0^+$, *and* $\frac{dy}{dx} \to 0^-$.
- *The curve has a horizontal tangent line at the critical point* $(3, 0)$.

Example 6.9.2 $x^2 + y^2 = xy + 3$ **show that** $(x - 2y)\frac{dy}{dx} = 2x - y$ **and deduce that the curve has vertical tangents at** $(2, 1)$ **and** $(-2, -1)$ **and horizontal tangents at** $(1, 2)$ **and** $(-1, -2)$. **Sketch the curve, showing these tangents.**

Solution:



Take the derivative of both sides with respect to x. consider y as a function of x and use the chain rule and product rule.

$x^2 + y^2 = xy + 3 \;\Rightarrow\; 2x + 2y\frac{dy}{dx} = y + x\frac{dy}{dx}$

$\therefore (x - 2y)\frac{dy}{dx} = 2x - y.$

substitution of $x = 2y$ *in the equation of the curve gives*

$x - 2y = 0 \;\Rightarrow\; 4y^2 + y^2 = 2y^2 = 3 \;\Rightarrow\; 3(y^2 - 1) = 0$

$\begin{cases} x = 2 \\ y = 1 \end{cases}$ *or* $\begin{cases} x = -2 \\ y = -1 \end{cases}$

In either case, $2x - y \neq 0$. *Hence as* $x - 2y = 0 \;\Rightarrow\; \frac{dy}{dx} = 0$ *and the curve has vertical tangents at* $(2, 1)$ *and* $(-2, -1)$.

Similarly, $2x - y = 0 \;\Rightarrow\; 3(x^2 - 1) = 0$

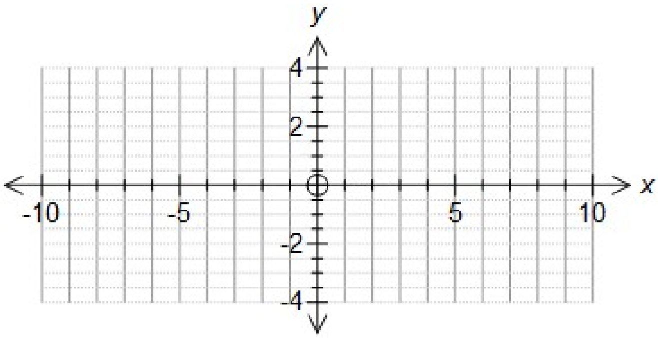
$\begin{cases} x = 1 \\ y = 2 \end{cases}$ *or* $\begin{cases} x = -1 \\ y = -2 \end{cases}$

In either case, $x - 2y \neq 0$. *Hence as* $2x - y = 0 \;\Rightarrow\; \frac{dy}{dx} = 0$ *and the curve has horizontal tangent at* $(1, 2)$ *and* $(-1, -2)$.

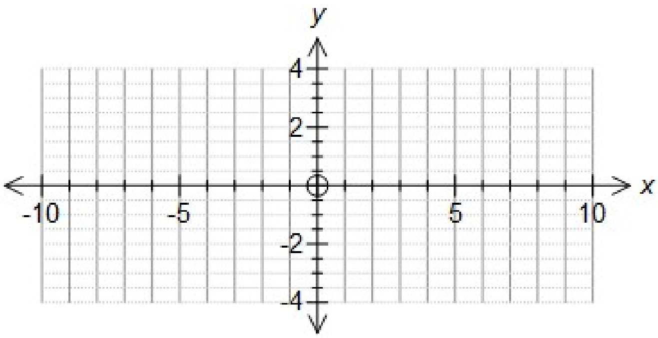
Clearly the curve is symmetric about $y = x$, *since the transformation* $y \leftrightarrow x$ *leave the Cartesian equation of the curve unchanged.*

Exercise 6.9.1

1. *Sketch (showing the critical points and stationary points) the graph of* $x^2 + y^2 = 4$.

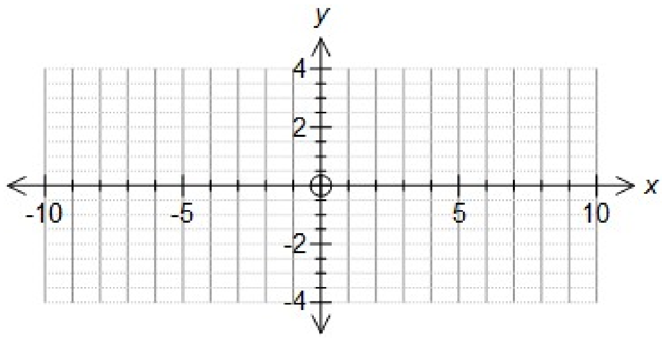


2. *Sketch (showing critical points and stationary points) the graph of* $x^2 - y^2 = 4$.

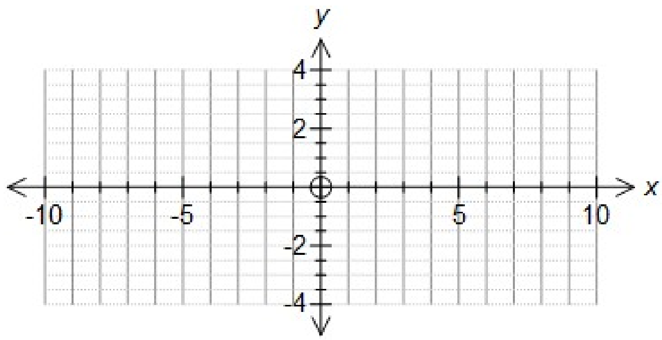


Exercise 6.9.2

1. *Sketch (showing critical points) the graph of* $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 1$.

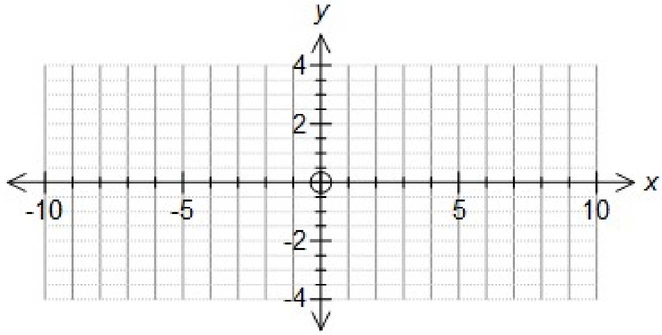


2. *Sketch (showing critical points and stationary points) the graph of* $x^2 + 4y^2 = 4$.

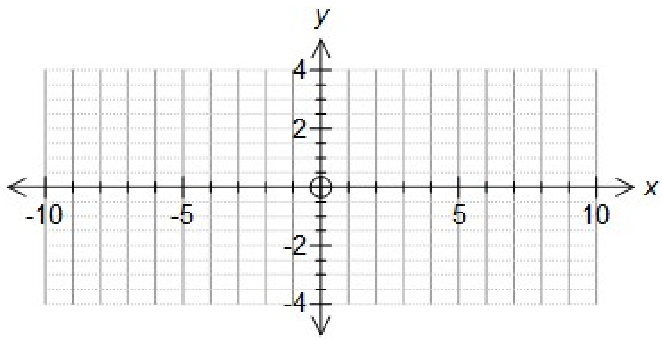


Exercise 6.9.3

1. *Sketch (showing critical points) the graph of* $x^2 - 4y^2 = 4$.



2. *Sketch (showing critical points and stationary points) the graph of* $x^3 + y^3 = 1$.

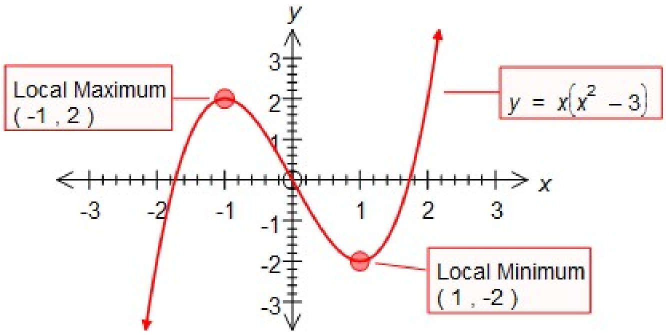


6.10 Using Graphs

Example 6.10.1

1. *Sketch the graph* $y = x(x^2 - 3)$*. showing the coordinates of the turning points and the intercepts on the coordinate axes. Deduce the set of values of k for which* $x(x^2 - 3) + k = 0$ *has exactly one real root.*

Solution:



$y = x^3 - 3x, \Rightarrow \frac{dy}{dx} = 3(x^2 - 1)$

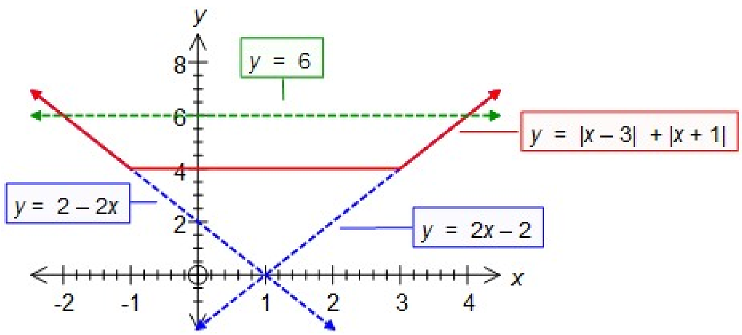
x		-1		0		1	
sign of $\frac{dy}{dx}$	+	0	-		-	0	+

If the graph $y = x(x^2 - 3)$ *is translated more than two units either upward or downward, the translated graph* $y = x(x^2 - 3) + k$ *will cut the x-axis exactly once. Hence* $x(x^2 - 3) + k = 0$ *will have exactly one real root if* $k > 2$ *or* $k < -2$*..*

Hence the set of values of k is $\{k : |k| > 2\}$*.*

2. *Sketch the graph* $y = |x - 3| + |x + 1|$ *and hence solve* $|x - 3| + |x + 1| > 6$*.*

Solution:

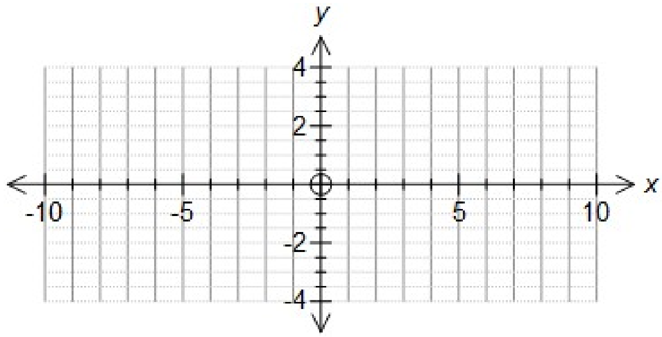


$$y = |x - 3| + |x + 1| = \begin{cases} = -(x-3) - (x+1) \\ = -(x-3) + (x+1) \\ = (x-3) + (x+1) \end{cases} = \begin{cases} 2 - 2x, & x < -1 \\ 4, & -1 \le x \le 3 \\ 2x - 2, & x > 3 \end{cases}$$

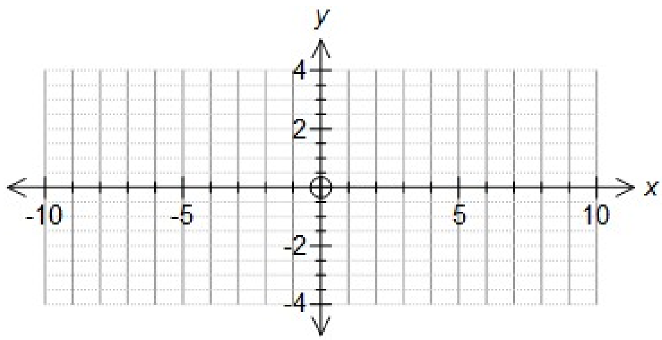
By inspection of the graph, $|x - 3| + |x + 1| > 6$ *for* $x < -2$ *or* $x > 4$*.*

Exercise 6.10.1

1. *Sketch the graph of* $y = x^3 - 12x$*. Use this graph to find the set of values of the real number k for which the equation* $x^3 - 12x + k = 0$ *has exactly one real root.*

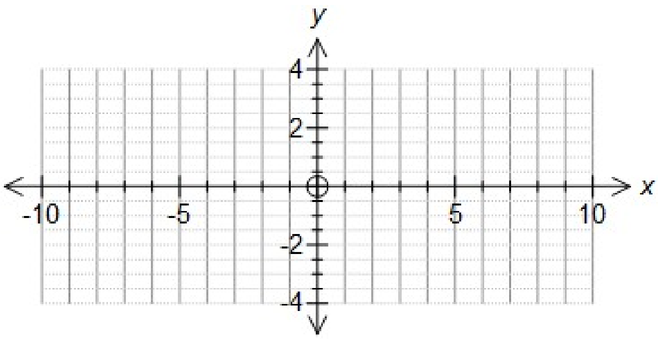


2. *A chord AB of a circle subtends an angle* θ *at the centre of the circle. If the perimeter of the minor segment is one-half the circumference of the circle, show that* $2\sin\frac{\theta}{2} = \pi - \theta$*. Solve the equation graphically.*



Exercise 6.10.2

1. *Sketch the graph of* $y = |x| - |x - 4|$ *use this graph to solve the inequality* $|x| - |x - 4| > 2$.



2. *Sketch the graph of* $y = \cos x$ *for* $0 \le x \le 2\pi$. *Use this grpha to solve the inequalities:*
(a) $\cos x \le \frac{1}{2}$ *for* $0 \le x \le 2\pi$, *(b)* $\cos x| \le \frac{1}{2}$ *for* $0 \le x \le 2\pi$