4 Unit Math Homework for Year 12

Student Name: ____________	Grade: ____________
Date: ____________	Score: ____________

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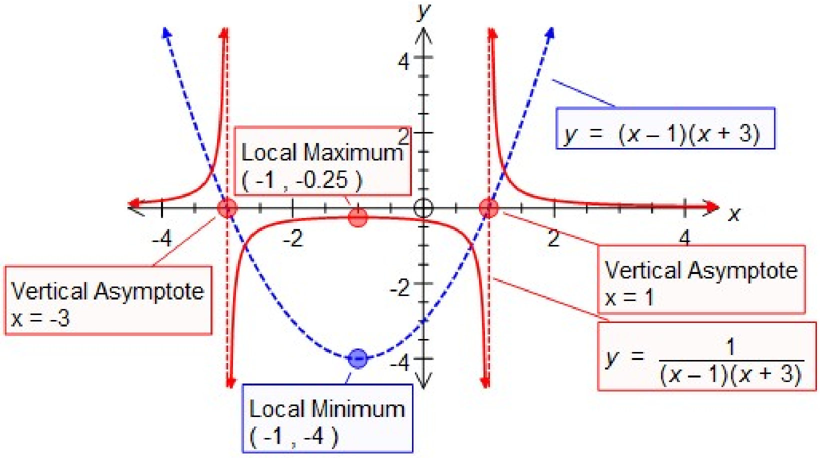
6 Topic 6 — Graphs Part 2

6.5 Reciprocal functions and division of ordinates

Consider the graphs $y = f(x)$ and $y = \frac{1}{f(x)}$, where $f(x) = (x-1)(x+3)$.

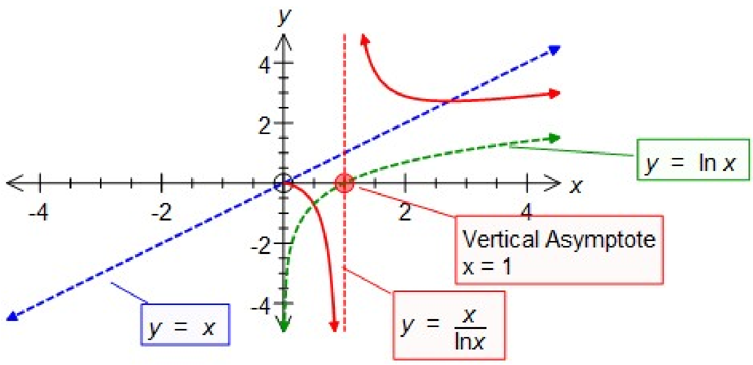


Features of the figures:

- $f(x), \frac{1}{f(x)}$ have the same sign.
- $y = f(x),\ y = \frac{1}{f(x)}$ intersect where $f(x) = \pm 1$.
- x-intercepts of $y = f(x)$ correspond to vertical asymptotes of $y = \frac{1}{f(x)}$.
- As $f(x) \to \infty,\ \frac{1}{f(x)} \to 0$.
- Minimum turning points of $y = f(x)$ correspond to maximum turning points of $y = \frac{1}{f(x)}$.

Example 6.5.1 Sketch the graphs $y = x$ and $y = \ln x$. Hence sketch $y = \frac{x}{\ln x}$.

Solution:



Features:

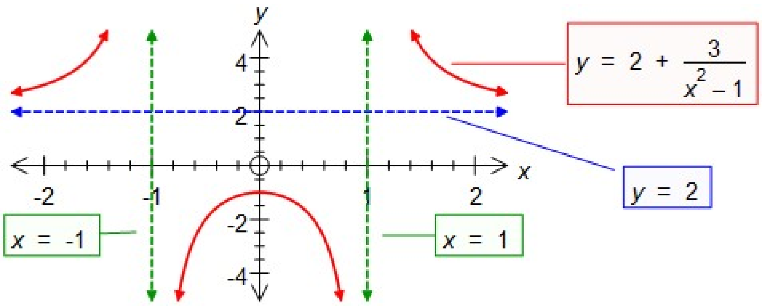
- *As* $x \to 0^+$, $\ln x \to -\infty$ *and* $\frac{x}{\ln x} \to 0^-$, *but* $x = 0$ *is exclude from the domain of* $y = \frac{x}{\ln x}$.
- *As* $x \to \infty, \ln x \to \infty$ *at a much slower rate than any power pf* x *and hence* $\frac{x}{\ln x} \to \infty$.
- *As* $x \to 1^-$, $\ln x \to 0^-$ *and* $\frac{x}{\ln x} \to -\infty$.

Definition: Functions of the form $\frac{P9x)}{Q(x)}$, where P(x) and Q(x) are polynomials, are called rational functions.
Whenever the degree of P(x) is equal to that of Q(x) or exceeds that of Q(x) by one, it is often easier to graph $y = \frac{P(x)}{Q(x)}$ after using the division transformation for polynomials.

Example 6.5.2

1. *Sketch the graphs* $y = \frac{2x^2+1}{x^2-1}$.

Solution:



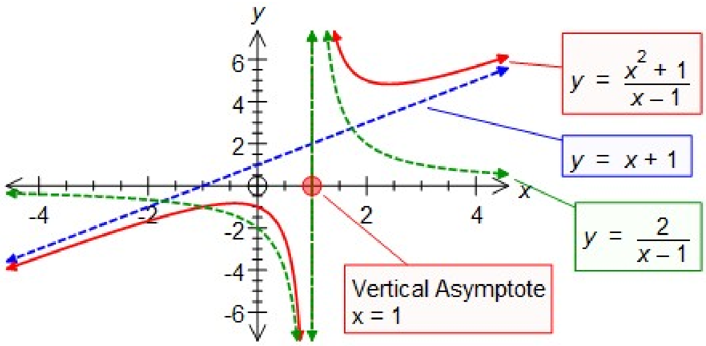
Degree $P(x) =$ *degree* $Q(x),\ y = \dfrac{2x^2+1}{x^2-1} = 2 + \dfrac{3}{x^2-1}$.

The graph $y = \dfrac{3}{x^2-1}$ *has been translted two units upward.*

$y = 2$ *is an asynmtote as* $x \to \infty$.

2. *Sketch the graphs* $y = \frac{x^2+1}{x-1}$.

Solution:

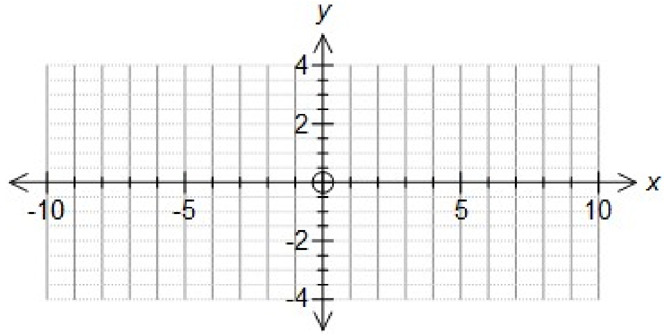


Degree $P(x) =$ *degree* $Q(x) + 1,\ y = \dfrac{x^2+1}{x-1} = (x+1) + \dfrac{2}{x-1}$.

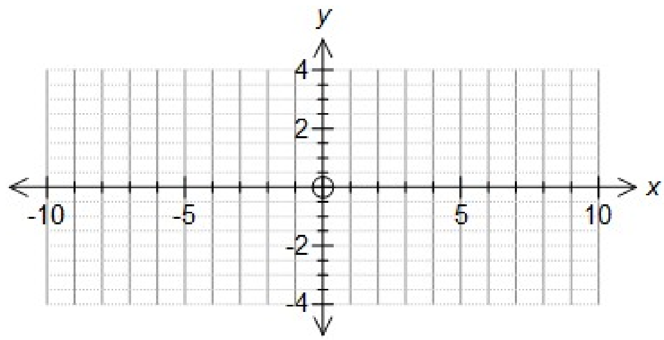
The graph has been constructed by addition of the ordinates of $y = x + 1$ *and* $y = \dfrac{2}{x-1}$.

$y = x + 1$ *is an asymptote as* $x \to \infty$.

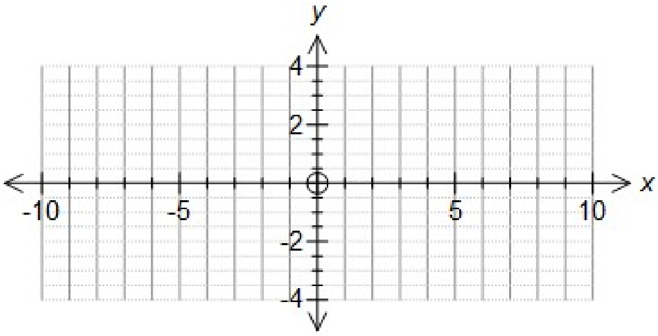
Exercise 6.5.1 Use the graph of $f(x) = 4 - x^2$ (an even function) to sketch the graph of $y = \frac{1}{f(x)}$. Is this the graph of an even function?



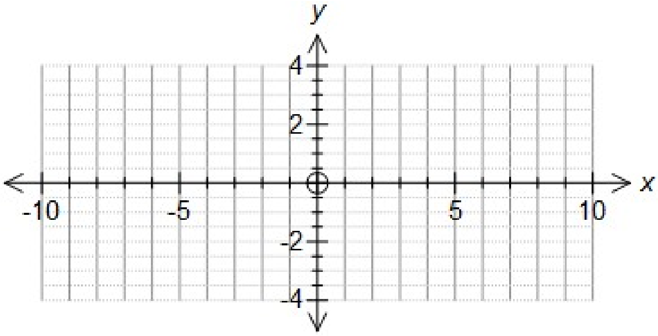
Exercise 6.5.2 Use the graph $y = x(x + 2)$ to sketch the graph of $y = \frac{1}{x(x+2)}$.



Exercise 6.5.3 Use the graphs of $y = x$ and $y = e^x$ to sketch the graph of $y = \frac{e^x}{x}$.

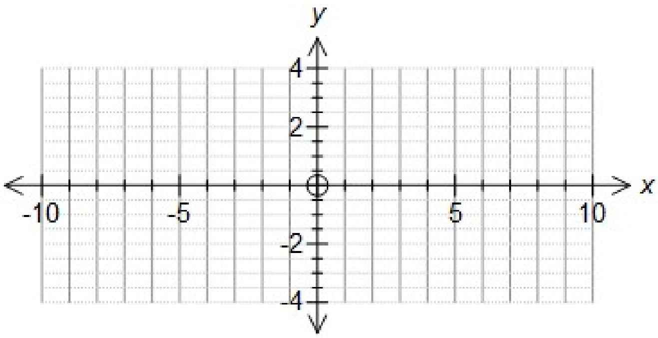


Exercise 6.5.4 Use the graphs of $y = x$ and $y = e^x$ to sketch the graph of $y = \frac{x}{e^x}$.

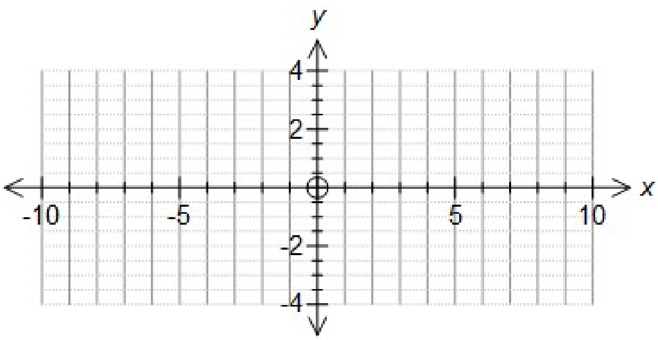


Exercise 6.5.5

1. *Show that* $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. *Hence sketch the graph of* $y = \frac{x^2}{x+1}$.



2. *Sketch the graph of* $y = \frac{x^2}{x^2-1}$.



6.6 Graphs of the form $y = [f(x)]^n$

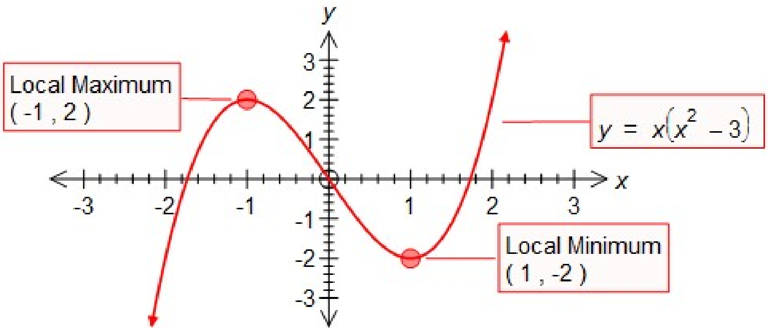
Definition: $y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1}. \times f'(x), (n > 1,\ n$ integral).
Hence the x-intercepts and the x-coordinates of stationary points of $y = f(x)$ give the stationary points of $y = [f(x)]^n$. the nature of these stationary points will depend on the signs of f and f' and on whether n is odd or even.

Example 6.6.1 Sketch $y = f(x)$, $y = [f(x)]^2$ **and** $y = [f(x)]^3$ **where** $f(x) = x(x^2 - 3)$.

1. $y = f(x)$, *where* $f(x) = x(x^2 - 3)$.

Solution: $f(x) = x(x^2 - 3) \Rightarrow f'(x) = 3(x^2 - 1)$. *Sign of* $f'(x)$

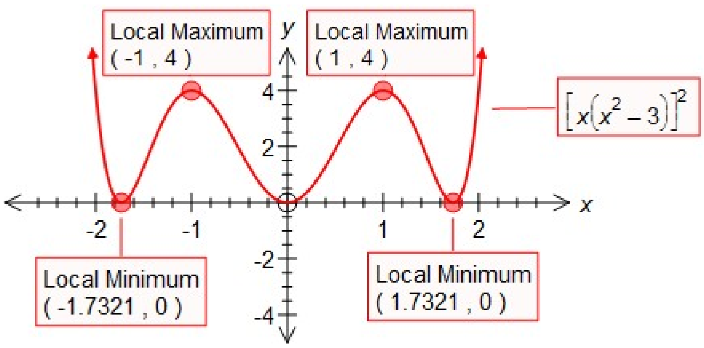
x	-2	-1	0	1	2
Sign of $f'(x)$	+	0	-	0	+



2. $y = f(x)$, *where* $f(x) = [x(x^2 - 3)]^2$.

Solution: $y = [f(x)]^2 \Rightarrow \frac{dy}{dx} = 2f(x) \times f'(x)$, *Sign of* $\frac{dy}{dx}$:

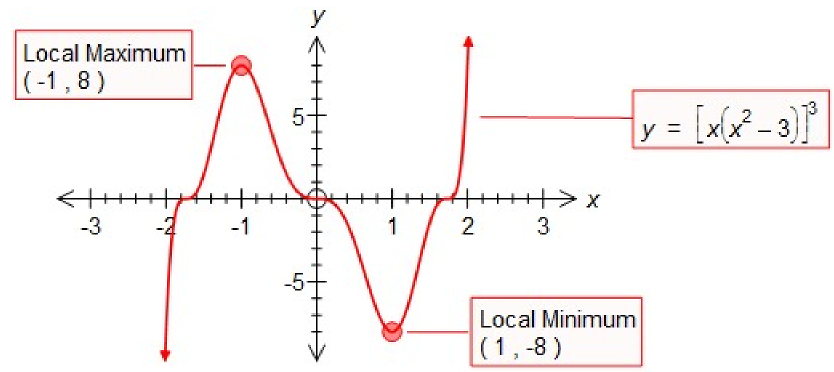
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
Sign of $f'(x)$	$-$	0	+	0	$-$	0	+	0	$-$	0	+



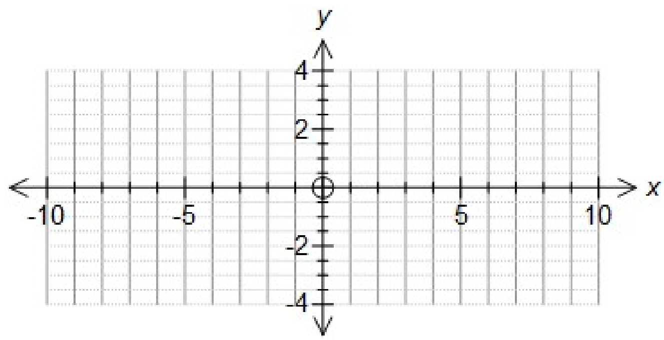
3. $y = f(x)$, *where* $f(x) = [x(x^2 - 3)]^3$.

Solution: $y = [f(x)]^3 \Rightarrow \frac{dy}{dx} = 3[f(x)^2]^2 \times f'(x)$, *Sign of* $\frac{dy}{dx}$:

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
Sign of $f'(x)$	$+$	0	$+$	0	$-$	0	$-$	0	$+$	0	$+$

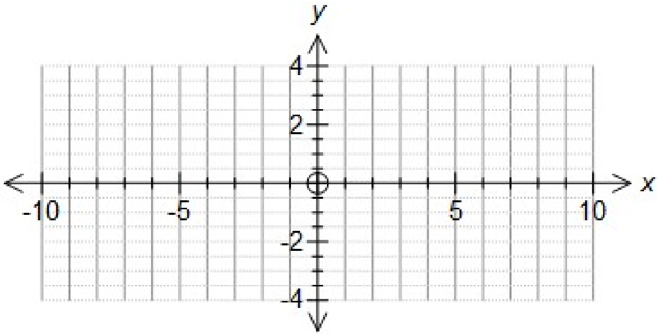


Exercise 6.6.1 Use the graph of $y = 3x - \frac{x^3}{4}$ **to sketch the graph of** $y = (3x - \frac{x^3}{4})^2$**.**

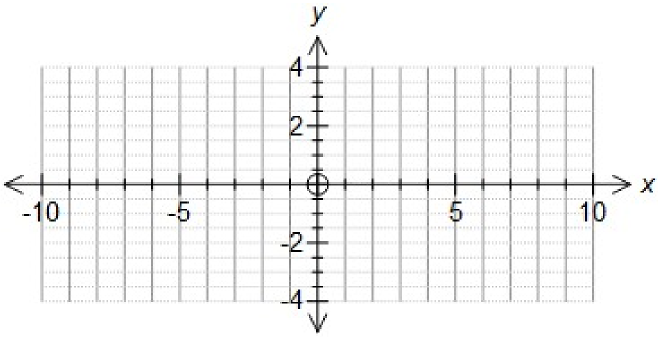


Exercise 6.6.2

1. *Use the graph of* $y = 1 - x^2$ *to sketch the graph of* $y = (1 - x^2)^2$.



2. *Use the graph of* $y = x^2 - 1$ *to sketch the graph of* $y = (x^2 - 1)^3$.



6.7 Graphs of the Form $y = \sqrt{f(x)}$

Definition: $y = \sqrt{f(x)} \;\Rightarrow\; \frac{dy}{dx} = \frac{f'(x)}{2\sqrt{f(x)}}$.

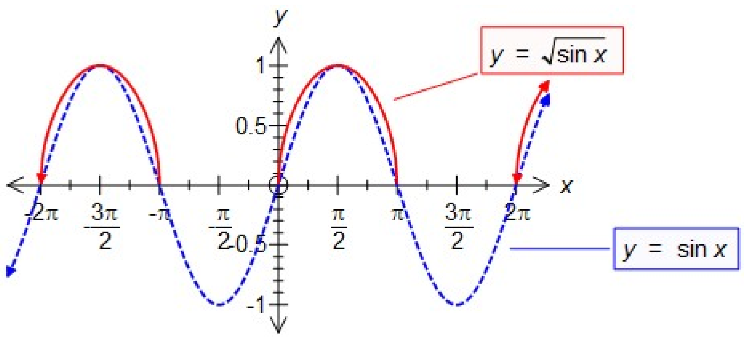
Hence x-intercepts of $y = f(x)$ give critical points of $y = \sqrt{f(x)}$ and the x-intercepts of stationary points of $y = f(x)$ give stationary points of $y = \sqrt{f(x)}$, provided $f(x) \neq 0$.

Clearly only those x-values in the domain of $f(x)$ for which $f(x) \geq 0$ are included in the domain of $y = \sqrt{f(x)}$.

Example 6.7.1

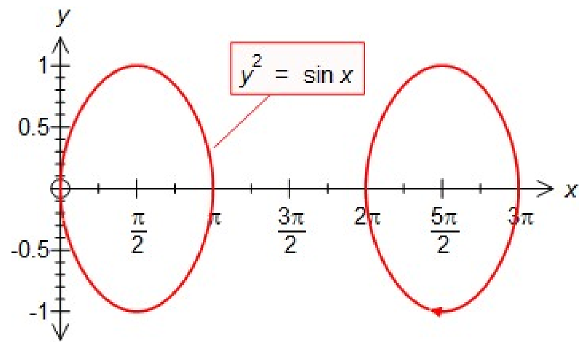
1. *On the same axes, sketch the graphs of* $y = \sin x$ *and* $y = \sqrt{(\sin x)}$.

Solution: $y = \sqrt{(\sin x)}$ *has vertical tangent lines at the critical points* $(n\pi, 0)$, n *integral. also* $0 < \sin x < 1 \;\Rightarrow\; \sqrt{(\sin x)} > \sin x$.



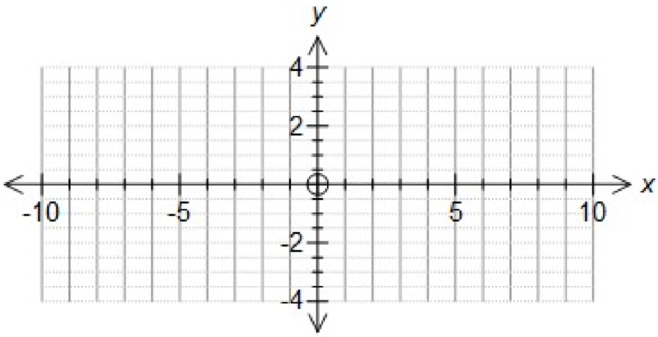
2. *On a new set of axes. sketch* $y^2 = \sin x$

Solution:

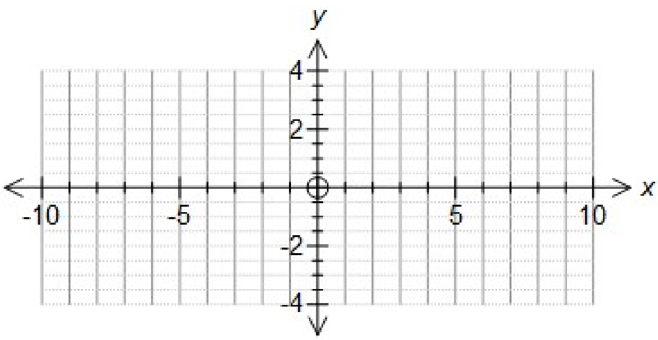


Exercise 6.7.1 For the function $f(x) = \cos x$ use the graph of $y = f(x)$ to sketch the graph of:

1. $y = \sqrt{f(x)}$,

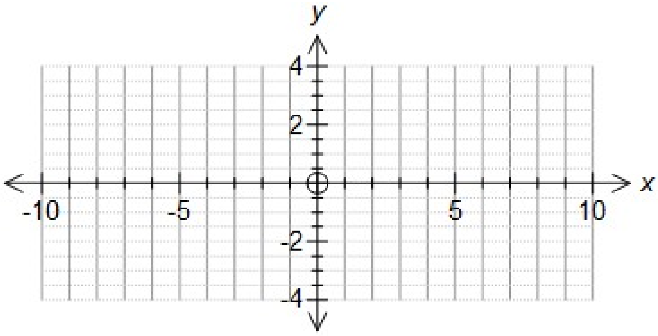


2. $y^2 = f(x)$.

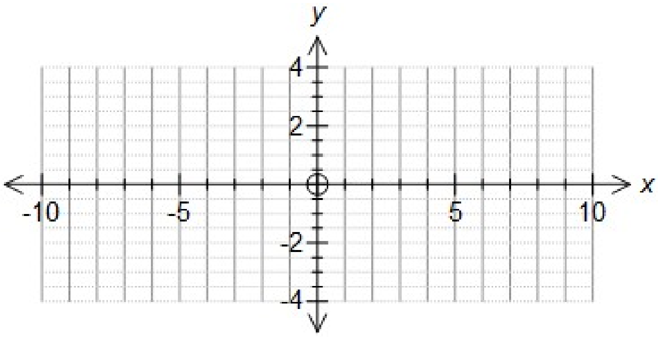


Exercise 6.7.2 For the function $f(x) = x^2 - 1$ use the graph of $y = f(x)$ to sketch the graphs of:

1. $y = \sqrt{f(x)}$.

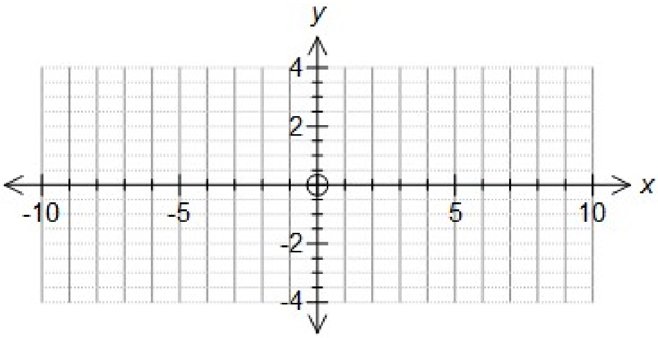


2. $y^2 = f(x)$.

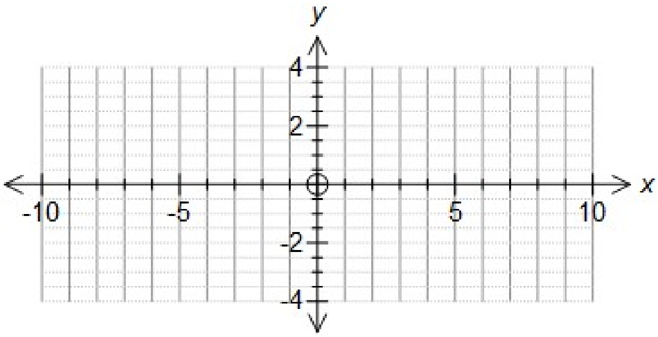


Exercise 6.7.3 Use the graph of $y = 4\cos x$ to sketch the graphs of:

1. $y = \sqrt{4\cos x}$.

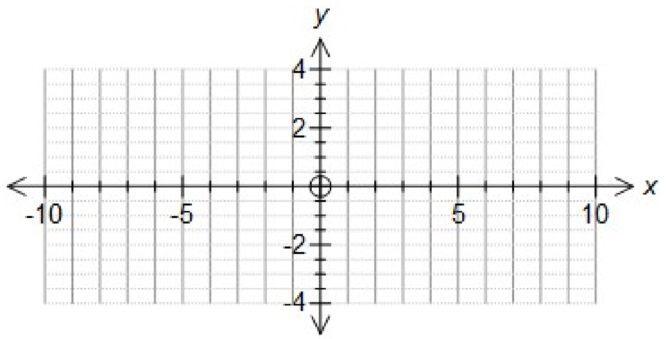


2. $y^2 = 4\cos x$.



Exercise 6.7.4 For the function $f(x) = 3x - \frac{x^3}{4}$, use the graph of $y = f(x)$ to sketch the graphs of:

1. $y = \sqrt{f(x)}$,



2. $y^2 = f(x)$.