4 Unit Math Homework for Year 12

Student Name: ______________	**Grade**: ______________
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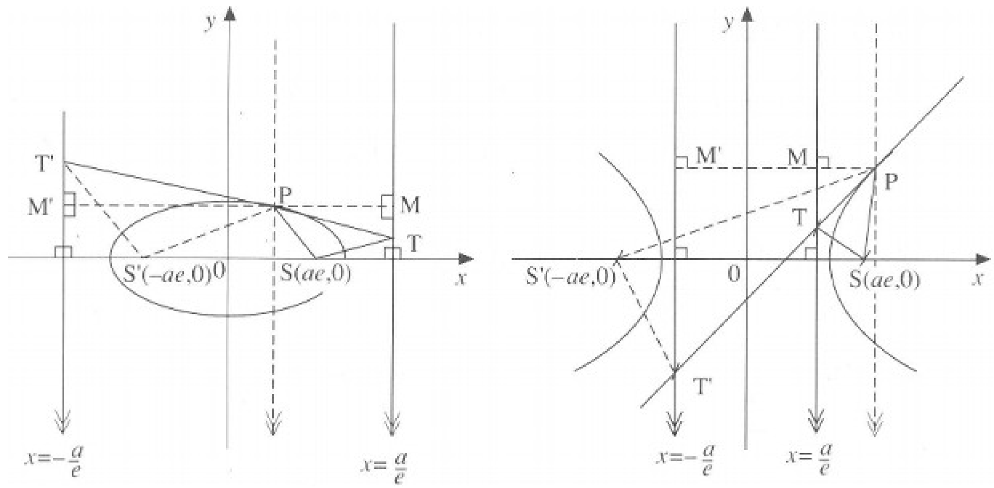
2 Topic 2 — Conics Part 4

2.8 Further Geometric Properties of the Ellipse and Hyperbola

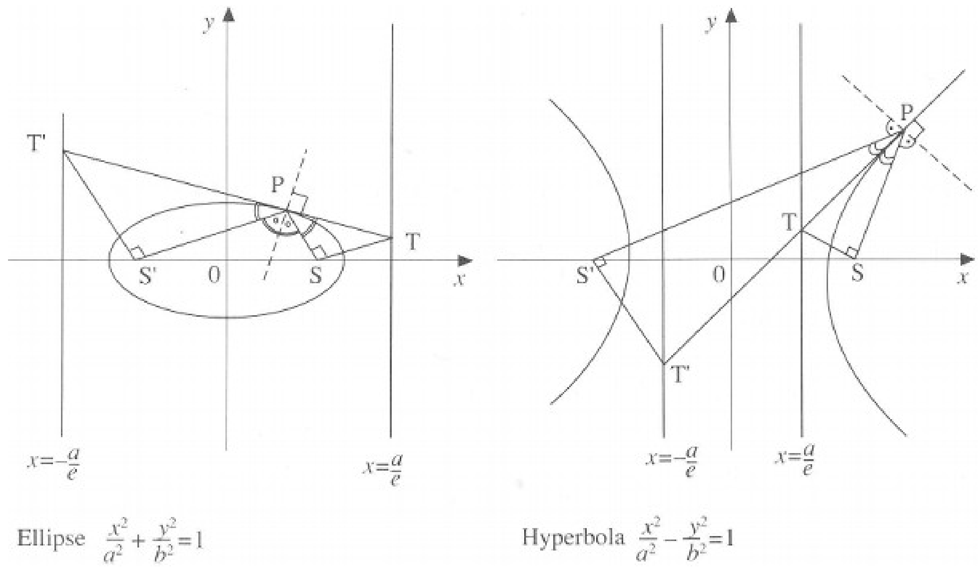
Definition:

1. The segment of the tangent to an ellipse or hyperbola between the point of the contact and the directrix subtends a right angle at the corresponding focus. ($PS \perp ST$)

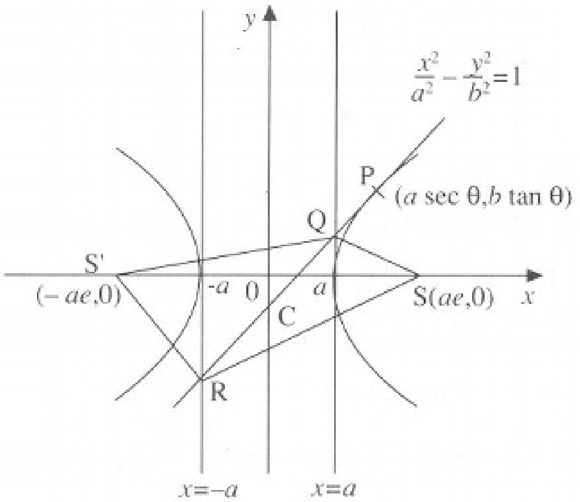


2. The tangent at a point P on the ellipse or hyperbola is equally inclines to the focal chords through P. ($\angle SPT = \angle S'PT'$)

3. The normal at a point P on the ellipse or hyperbola is equally inclined to the focal chords through P.



Example 2.8.1 $P(a\sec\theta,\, b\tan\theta)$ **lies on the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus. Deduce that if P is the point** $5,\ \frac{4}{3}$ **on the hyperbola** $\frac{x^2}{9} - y^2 = 1$**, with foci S and S', then Q, S, S, S' are concyclic and find the equation of the circle through these points.**



Solution: *Let the tangent at P meet* $x = a,\ x = -a$ *in the* $Q,\ R$ *respectively.*

Let QR *meet the y-axis in* C*.*

Tangent PR has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Hence Q has coordinates $\left[a,\ \frac{b}{\tan\theta}(\sec\theta - 1)\right]$ *and R has coordinates* $\left[-a,\ -\frac{b}{\tan\theta}(\sec\theta + 1)\right]$

Gradient QS*.gradient* $RS = \frac{b(\sec\theta - 1}{a\tan\theta(1-e)} \times \frac{-b(sec\theta+1}{-a\tan\theta(1+e)}$

$= \frac{b^2}{a^2(1-e^2)} \times \frac{(\sec^2\theta - 1)}{\tan^2\theta}$.

Then $b^2 = a^2(e^2 - 1) \ \Rightarrow$ *gradient* $QS\times$ *gradient* $RS = -1,\ \therefore QS \perp RS$

Similarly, replacing e *by* $-e, QS' \perp RS'$.

Hence QR subtends angles of 90° *at each of S and S', and Q, S, R, S' are noncyclic, with QR the diameter of the circle through the points.*

The y-axis is the perpendicular bisector of the chord SS',

Hence the centre of this circle is the point C where the diameter QR meets the y-axis.

If $P(5,\ \frac{4}{3}$ *lies in the hyperbola* $\frac{x^2}{9} - y = 1$*, then QR has equation* $\frac{5x}{9} - \frac{4y}{3} = 1$ *and meets the y-axis in* $C\left(0,\ \frac{3}{4}\right)$.

Also $b^2 = a^2(e^2 - 1)$ *gives* $e = \frac{10}{9}$*, and S ahs coordinates* $(\sqrt{10},\ 0)$.

Hence $CS^2 = \frac{169}{16}$ *and the circle through* $Q,\ R,\ S,\ S'$ *has equation* $x^2 + \left(y + \frac{3}{4}\right)^2 = \frac{169}{16}$.

2.9 The Rectangular Hyperbola

2.9.1 Cartesian Equation of the Rectangular Hyperbola

Definition: A hyperbola is called rectangular if its asymptotes meet at right angles.

The asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ have equations $y = \pm\frac{b}{a}x$.

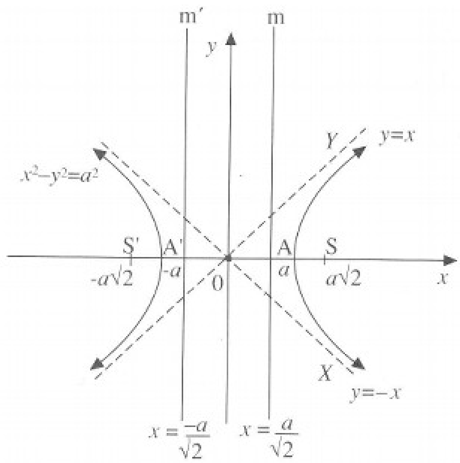
Hence asymptotes perpendicular $\Rightarrow$ $(\frac{b}{a}) \times (-\frac{b}{a} = -1), \Rightarrow a = b$.

Then $b^2 = a^2(e^2 - 1), \Rightarrow e = \sqrt{(1 + \frac{b^2}{a^2})}, \Rightarrow \therefore e = \sqrt{2}$.

Hence a rectangular hyperbola, with major axis along the x-axis,

has equation $x^2 - y^2 = a^2$,

eccentricity $e = \sqrt{2}$ and asymptotes $y = \pm x$.

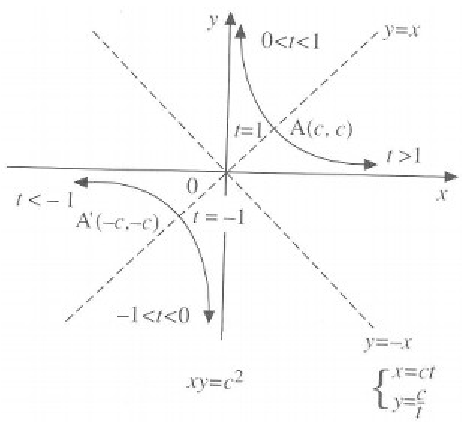


2.9.2 Parametric Equations of the Rectangular Hyperbola

Definition: The standard parametric from of the rectangular hyperbola $xy = c^2$ is

$x = ct$ and $y = \frac{c}{t}$,

where the value of parameter t depends on the position of $P(ct, \frac{c}{t})$ on the curve as in the figure shown below:



Exercise 2.9.1

1. *For the rectangular hyperbola* $xy = 4$*, find the parametric equation.*

2. *For the rectangular hyperbola* $x = 4t,\ y = \frac{4}{t}$*, find the Cartesian equation.*

3. *Find the parametric equation of the rectangular hyperbola* $xy = 25$.

4. *Find the Cartesian equation of the rectangular hyperbola* $x = 3t,\ y = \frac{3}{t}$.

2.9.3 Equation of a Chord of the Hyperbola $xy = c^2$

Cartesian form:

Let $P(x_1,\, y_1),\ Q(x_2,\, y_2)$ lie on the hyperbola $xy = c^2$.

The gradient of PQ is $\frac{y_2-y_1}{x_2-x_1} = \frac{1}{x_2-x_1}\left(\frac{c^2}{x^2} - \frac{c^2}{x_1^2}\right) = \frac{-c^2}{x_1x_2}$

$\therefore$ chord PQ has equation $c^2x + x_1x_2y = k,\ k$ is constant.

$\therefore k = c^2(x_1 + x_2)$.

The chord from $P(x_1,\, y_1)$ to $Q(x_2,\, y_2)$ on $xy = c^2$

has equation $c^2x + x_1x_2y = c^2(x_1 + x_2)$.

Parametric form:

Let $P\left(cp,\, \frac{c}{p}\right),\ Q\left(cq,\, \frac{c}{q}\right)$ lie on the hyperbola $xy = c^2$.

The gradient of PQ is $\frac{c(\frac{1}{q}-\frac{1}{p})}{c(q-p)} = -\frac{1}{pq}$

$\therefore$ chord PQ has equation $x + pqy = k,\ k$ is constant.

$P(cp,\ \frac{c}{p})$ lies on $PQ \ \Rightarrow \therefore k = c(p+q)$.

The chord from $P(cp,\, \frac{c}{p})$ to $Q(cq,\, \frac{c}{q})$ on $xy = c^2$

has equation $x + pqy = c(p+q)$.

Example 2.9.1 The points $P(cp,\, \frac{c}{p})$ and $Q(cq,\, \frac{c}{q})$ lie on the rectangula hyperbola $xy = c^2$. The chord PQ subtends a right angle at the another point $R(cr,\, \frac{c}{r})$ on the hyperbola. Show the the normal at R is parallel to PQ.

Solution: *The gradient of PR is* $\frac{c(\frac{1}{p}-\frac{1}{r})}{c(p-r)} = -\frac{1}{pr}$,

the gradient of QR is $\frac{c(\frac{1}{q}-\frac{1}{r})}{c(q-r)} = -\frac{1}{qr}$.

Therefore $PQ \perp QR,\ \Rightarrow$ *gradient* $PR\times$ *gradient* $QR = -1$,

$\therefore \frac{1}{pqr^2} = -1 \ \Rightarrow \ r^2 = -\frac{1}{pq}$.

The normal at the point $R(cr,\, \frac{c}{r})$ *has gradient of* r^2,

the gradient of PQ is $\frac{c(\frac{1}{p}-\frac{1}{q}}{c(P-q)} = -\frac{1}{pq}$.

Since $r^2 = -\frac{1}{pq}$, *then gradient of the normal at R equal to gradient of PQ.*

Thus the normal at the point R is parallel to the chord PR.

2.9.4 Equations of Tangents and Normals to the hyperbola $xy = c^2$

Cartesian from of equation of tangent:

By implicit differentiation, $xy = c^2, \Rightarrow y + x\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{y}{x}$

Gradient of tangent at $P(x_1, y_1)$ on the hyperbola is $-\frac{y_1}{x_1}$

$\therefore$ tangent at P has equation $y_1 x + x_1 y = k,\ k$ is constant.

By P lies on tangent $\therefore k = 2x_1y_1 = 2c^2$.

The tangent to $xy = c^2 at P(x_1, y_1)$ has equation $xy_1 + yx_1 = 2c^2$

Cartesian from of equation of normal:

Also the normal at P (x_1, y_1) and equation $y - y_1 = \frac{x_1}{y_1}(x - x_1)$.

The normal to $xy = c^2$ at $P(x_1, y_1)$ has equation $xx_1 - yy_1 = x_1^2 - y^2$.

Parametric form of equation of tangent:

$x = ct, \Rightarrow \frac{dx}{dt} = c$, and $y = \frac{c}{t}, \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$,

$\therefore \frac{dy}{dx} = -\frac{1}{t^2}$

Gradient of tangent at $P(cp, \frac{c}{p})$ on the hyperbola is $-\frac{1}{p^2}$,

$\therefore$ tangent at P has equation $x + p^2 y = k$, k is constant.

But P lies on tangent $\therefore k = 2cp$.

The tangent to $xy = c^2$ at $P\left(cp, \frac{c}{p}\right)$ has equation $x + p^2y = 2cp$.

Parametric form of equation of normal:

Also the normal at $P\left(cp, \frac{c}{p}\right)$ has gradient P^2 and equation $y - \frac{c}{P} = p^2(x - cp)$.

The normal to $xy = c^2$ at $P\left(cp, \frac{c}{p}\right)$ has equation $px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right)$.

Example 2.9.2 Find the equation of the tangent and the normal to the rectangular hyperbola $x = 2t, y = \frac{2}{t}$ at the point where $t = 4$.

Solution: *For the hyperbola $x = 2t, y = \frac{2}{t}$, where we have c = 2.*

Hence the tangent to the hyperbola $x = 2t, y = \frac{2}{t}$ at point where $t = 4$

has equation $x + t^2y = 2ct, \Rightarrow x + 16y = 16$

and the normal has equation $tx - \frac{1}{t}y = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow 32x - 2y = 255$.

Exercise 2.9.2

1. *Find the equations of the tangent and the normal to the rectangular hyperbola* $xy = 12$ *at the point (-3, -4).*

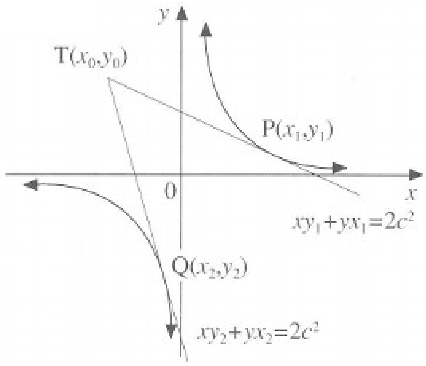
2. *Find the equations of the tangent and the normal to the rectangular hyperbola* $x = 3t$, $y = \frac{3}{t}$ *at the point* $t = -1$.

3. *Find the equation of chord of contact of tangent from the point (1, -2) ro the hyperbola* $xy = 6$.

4. *Find the equation of the tangent and the noram to the rectangular hyperbola* $xy = 8$ *at the point* $(4, 2)$.

2.9.5 Equation of the Chord of Contact of Tangent from an External Point

Definition: T lies on the tangent PT and QT, $\therefore$ $x_0y_1 + y_0x_1 = 2x^2$ and $x_0y^2 + y_0x_2 = 2c^2$.
Hence $P(x_1,\, y_1)$ and $Q(x_2,\, y_2)$ both satisfy $x_0y + y_0x = 2c^2$.
$\therefore$ this linear equation must be the equation of PQ.



The chord contact of tangent from an external point $T(x_0,\, y_0)$ to the hyperbola $xy = c$ has equation $xy_0 + yx_0 = 2c^2$.

Example 2.9.3 Find the equation of the chord of contact of tangent from the point (2, 1) to $xy = 10$.

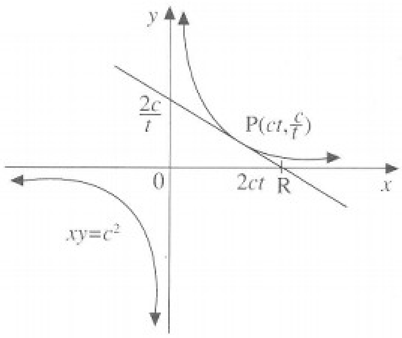
Solution: *For the hyperbola $xy = 10$, we have $c^2 = 10$.*
Hence the chord of contact of tangents from the point $T(x_0,\, y_0) = T(2,\, 1)$
to the hyperbola $xy = 10$ has equation $xy_0 + yx_0 = 2c^2 \;\Rightarrow\; x + 2y = 20$.

Exercise 2.9.3 Find the equation of the chord of contact of tangents from the point $(-1,\, -3)$ to the rectangular hyperbola $xy = 4$. Hence find the coordinated of their points of contact and the equations of these tangents.

2.9.6 Further geometric properties of the rectangular hyperbola

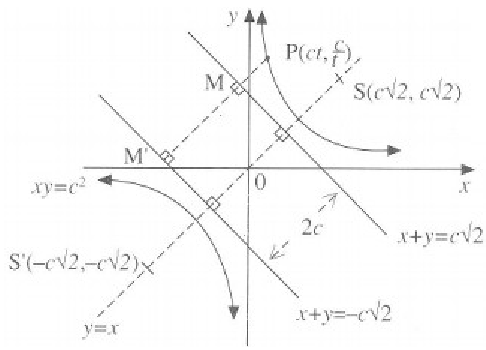
Definition:

1. The area of the triangle bounded by a tangent and the asymptotes is a constant.
   Let the tangent at $P(ct, \frac{c}{t}$ on $xy = c^2$ meet the x-axes and y-axes in R and T respectively.
   The tangent has equation $x + t^2y = 2ct$. At $T, x = 0,\ y = \frac{2c}{t}$.
   $\therefore\ Area\,\triangle OTR = \frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2.$



2. The length of the intercept cut off a tangent by the asymptotes is twice the distance from the point of the intersection of the asymptotes to the point of the contact of the tangent.
   $TR^2 = (\frac{2c^2}{t}) + (2ct)^{=}4c^2(t^2 + \frac{1}{t^2})$, and $OP^2(ct)^2 + (\frac{c}{t})^2 = c^2(t^2 + \frac{1}{t^2})$.
   $TR^2 = 4OP^2,\ \Rightarrow \therefore TR = 2OP.$

3. The product of the focal distance of a point P on a rectangular hyperbola is equal to the square of the distance from P to the point of intersection of the asymptotes.
   Let $xy = c^2$ have foci S and S', and directrices m and m'. Let M, M' be the feet of the perpendiculars from $P(ct, \frac{c}{t}$ to m, m' respectively.
   Then $PS \times PS' = ePM \times ePM' = 2PM \times PM'\ (since\, e = \sqrt{2})$,
   $\therefore\ PS \times PS' = 2 \times \frac{|ct+\frac{c}{t}-\sqrt{2}c|}{\sqrt{2}} \times \frac{|ct+\frac{c}{t}+\sqrt{2}c|}{\sqrt{2}} = |(ct + \frac{c}{t})^2| - (\sqrt{2}c)^2$
   $\therefore\ PS \times PS' = (ct)^2 + (\frac{c}{t})^2 = OP^2.$



Example 2.9.4 For the rectangular hyperbola $xy = 16$, find (a) the eccentricity; (b) the coordinates if the foci; (c) the equation s of the directrices; (d) the equations of the asymptotes. Sketch the hyperbola.

Solution: *For the hyperbola $xy = 16$ we have $c^2 = 16 \;\Rightarrow\; c = 4$.*

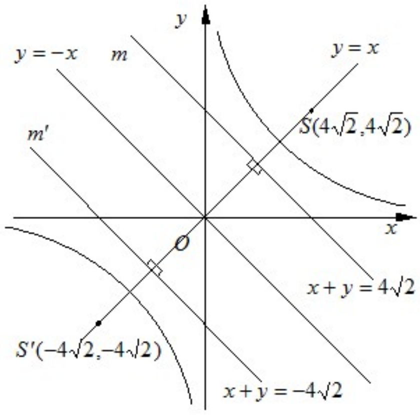
Hence the hyperbola $xy = 16$ has eccentricity $e = \sqrt{2}$

Foci: $S(c\sqrt{2},\, c\sqrt{2}), \;\Rightarrow\; S(4\sqrt{2},\, 4\sqrt{2})$

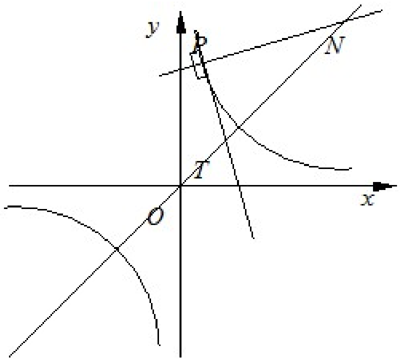
and $S'(-c\sqrt{2},\, -c\sqrt{2}), \;\Rightarrow\; S'(-4\sqrt{2},\, -4\sqrt{2})$,

Dirctires: $x + y = \pm c\sqrt{2}, \;\Rightarrow\; x + y = \pm 4\sqrt{2}$,

Asymptotes: $x = 0$ *and* $y = 0$.

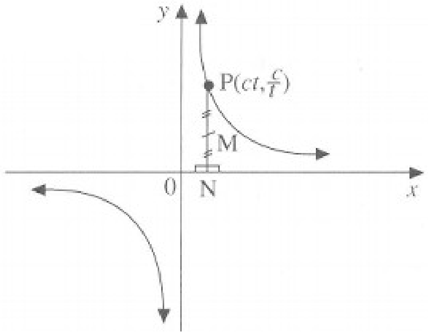


Exercise 2.9.4 The point $P(ct,\, \frac{c}{t})$, where t = -1 lies on the rectangular hyperbola $xy = c^2$. The tangent and normal at P meet the line $y = x$ at T and N respectively. Show that $OT \times ON = 4c^2$.



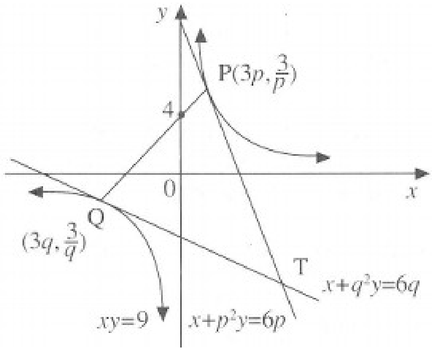
2.9.7 Locus Problem and the Rectangular Hyperbola

Example 2.9.5 $P(ct,\, \frac{c}{t})$ **is a point on the rectangular hyperbola** $xy = c^2$**. N is the foot of the perpendicular from P to the x-axis and M is the midpoint of** PN**. Find the Cartesian equation of the locus of M as the position of P varies. and describe this locus geometrically.**



Solution: *M has coordinates* $(ct,\, \frac{c}{2t})$.
Hence the locus of M has parametric equations $x = ct$,
and $y = \frac{c}{2t}$ *and Cartesian equation* $xy = \frac{1}{2}c^2$.
M traces a rectangular hyperbola as the position of P varies.

Example 2.9.6 $P(3p,\, \frac{3}{p})$ **and** $Q(3q,\, \frac{3}{q})$ **are point on different branches of the hyperbola** $xy = 9$**. Find the coordinates of the point of intersection T of the tangents at** P **and** Q**. Find the locus of** T **if the positions of** P **and** Q **very so that the chord** PQ **passes through** $(0,\, 4)$**.**



Solution: *T lies on the tangent at P and Q.* $\Rightarrow$ *At T,* $\begin{cases} x + p^2y = 6p & (1) \\ x + q^2y = 6q & (2) \end{cases}$

$(1) - (2) \;\Rightarrow\; (p^2 - q^2)y = 6(p - q),\; p \neq q \;\Rightarrow \therefore\; y = \frac{6}{p+q}$

Then $(1) \;\Rightarrow\; x = \frac{6pq}{p+q},\; \Rightarrow$ *T has coordinates* $\left(\frac{6pq}{p+q},\, \frac{6}{p+q}\right)$.

If $(0,\, 4)$ *lies on chord PQ, with equation* $x + pqy = 3(p + q)$,

then $4pq = 3(p + q)$ *and T is* $T\left(\frac{9}{2},\, \frac{9}{2pq}\right)$.

P, Q on different branches $\Rightarrow\; pq < 0,\; \Rightarrow \therefore$ *locus of T is* $x = \frac{9}{2},\, y < 0$

Exercise 2.9.5

1. *P and Q are variable points on the rectangular hyperbola $xy = 9$. The tangents at P and Q meet at R. If PQ passes through the point $(6,\, 2)$. find the equation of the locus of R.*

2. *The point $P(ct,\, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that the area of $\triangle YOX$ is independent of t.*

3. *On the rectangular hyperbola $xy = c^2$ there are variable points P and Q. The tangents at P and Q meet at R. Find the equation of the locus of R if PQ passes through the point $(a,\, 0)$.*

4. *The point $P(ct,\, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that PX = PY.*

Exercise 2.9.6

1. *The point* $P(ct,\, \frac{c}{t})$ *lies on the rectangular hyperbola* $xy = c^2$. *Show that the normal at P cuts the hyperbola again at the point Q with coordinates* $(-\frac{c}{t^3},\, -ct^3)$. *Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again.*

2. *The point* $P(ct,\, \frac{c}{t})$ *lies on the rectangular hyperbola* $xy = c^2$. *The normal at P meets the x-axis at A and the tangent at P meets the y-axis at B. M is the midpoint of AB. Find the equation of the locus of M as P moves on the hyperbola.*