4 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

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2 Topic 2 — Conics Part 3

2.6 Tangents and Normals to the Ellipse and Hyperbola

2.6.1 Equations of Tangents to the Ellipse

Cartesian Form: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by implicit differentiation $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore$ the gradient of the tangent at $P(x_1,y_1)$ is $-\frac{b^2}{a^2}\frac{x_1}{y_1}$.

$\therefore$ the equation of the tangent at P is: $y-y_1=-\frac{b^2}{a^2}\frac{x_1}{y_1}(x-x_1) \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$

$\therefore$ $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is the tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1,\ y_1)$.

Parametric Form: $x=a\cos\theta, \Rightarrow \frac{dx}{d\theta}=-a\sin\theta,\quad y=b\sin\theta, \Rightarrow \frac{dy}{d\theta}=b\cos\theta$

$\therefore$ the gradient of the tangent at $P(a\cos\phi,\ b\sin\phi)$ is $-\frac{b\cos\phi}{a\sin\phi}$,

$\therefore$ the equation of the tangent at P is $y-b\sin\phi=-\frac{b\cos\phi}{a\sin\phi}(x-a\cos\phi)$

$\Rightarrow \frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=\cos^2\phi+\sin^2\phi=1.$

$\therefore$ $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$ is the tangent to the ellipse

$x=a\cos\theta,\ y=b\sin\theta$ at $P(a\cos\phi,\ b\sin\phi)$.

2.6.2 Equations of Normals to the Ellipse

Cartesian Form: Also the gradient of the normal at P is $\frac{a^2}{b^2}\frac{y_1}{x_1}$,

$\therefore$ the equation of the normal at P is $y-y_1=\frac{a^2}{b^2}\frac{y_1}{x_1}(x-x_1)$

$\therefore$ $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ is the the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1,\ y_1)$.

Parametric From: Also the gradient of the normal at P is $\frac{a\sin\phi}{b\cos\phi}$,

$\therefore$ the equation of the normal at P is $y-b\sin\phi=\frac{a\sin\phi}{b\cos\phi}(x-a\cos\phi)$.

$\therefore$ $\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$ is the normal to the ellipse

$x=a\cos\theta,\ y=b\sin\theta$ at $P(a\cos\phi,\ b\sin\phi)$.

Example 2.6.1 Find the equations of the tangent and normal to the ellipse $3x^2+4y^2=48$ **at the point** $(2,\ -3)$.

Solution: $3x^2 + 4y^2 = 48 \;\Rightarrow\; \dfrac{x^2}{16} + \dfrac{y^2}{12} = 1.$

The tangen to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$ *at the point (2, -3) has equation*

$$\frac{2x}{16} + \frac{(-3y)}{12} = 1 \;\Rightarrow\; x - 2y = 8.$$

The normal to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$ *at the point (2, -3) has equation*

$$\frac{16x}{2} - \frac{12y}{(-3)} = 16 - 12, \; 2x + y = 1.$$

2.6.3 Equations of Tangents to the Hyperbola

Cartesian Form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by implicit differentiation $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore$ the gradient of the tangent at $P(x_1, y_1)$ is $\frac{b^2}{a^2}\frac{x_1}{y_1}$.

$\therefore$ the equation of the tangent at P is: $y - y_1 = \frac{b^2}{a^2}\frac{x_1}{y_1}(x - x_1) \Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$

$\therefore \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is the tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$.

Parametric Form: $x = a\sec\theta, \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad y = b\tan\theta, \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$

$\therefore$ the gradient of the tangent at $P(a\sec\phi, b\tan\phi)$ is $fracb\sec\phi a\tan\phi$,

$\therefore$ the equation of the tangent at P is $y - b\tan\phi = \frac{b\sec\phi}{a\tan\phi}(x - a\sec\phi)$

$\Rightarrow \frac{x\sec\phi}{a} + \frac{y\tan\phi}{b} = \sec^2\phi - \tan^2\phi = 1.$

$\therefore \frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1$ is the tangent to the hyperbola

$x = a\sec\theta, y = b\tan\theta$ at $P(a\sec\phi, b\tan\phi)$.

2.6.4 Equations of Normal to the Hyperbola

Cartesian Form: Also the gradient of the normal at P is $-\frac{a^2}{b^2}\frac{y_1}{x_1}$,

$\therefore$ the equation of the normal at P is $y - y_1 = -\frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$

$\therefore \frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$ is the the normal to the hypebola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$.

Parametric From: Also the gradient of the normal at P is $\frac{-a\tan\phi}{b\sec\phi}$,

$\therefore$ the equation of the normal at P is $y - b\tan\phi = \frac{-a\tan\phi}{b\sec\phi}(x - a\sec\phi)$.

$\therefore \frac{ax}{\sec\phi} + \frac{by}{\tan\phi} = a^2 + b^2$ is the normal to the hyperbola

$x = a\sec\theta, y = b\tan\theta$ at $P(a\cos\phi, b\sin\phi)$.

Example 2.6.2 Find the equation of tangent and the normal to the hyperbola $9x^2 - 2y^2 = 18$ at the point $(2, -3)$.

Solution: $9x^2 - 2y^2 = 18 \Rightarrow \frac{x^2}{2} - \frac{y^2}{9} = 1.$

The tangent to the hyperbola $\frac{x^2}{2} - \frac{y^2}{9} = 1$ *at the point (2, -3) has equation*

$$\frac{2x}{2} - \frac{-3y}{9} = 1 \Rightarrow 3x + y = 3.$$

The normal to the hyperbolar $\frac{x^2}{2} - \frac{y^2}{9} = 1$ *at the point (2, -3) has equation*

$$\frac{2x}{2} + \frac{9y}{-3} = 2 + 9 \Rightarrow x - 3y = 11.$$

Exercise 2.6.1

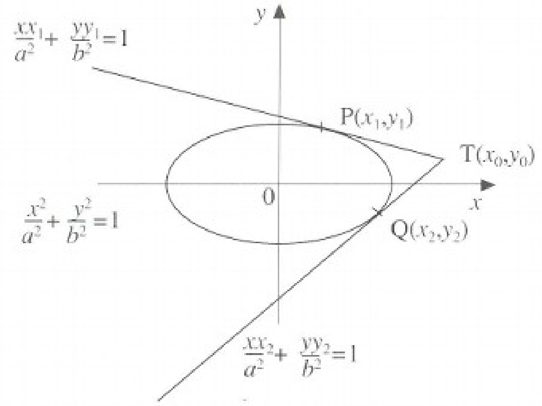
1. *Find the equation of the tangent and the normal to ellipse* $x = 4\cos\theta,\ y = 2\sin\theta$ *at the point where* $\theta = -\frac{\pi}{4}$.

2. *Find the euation of the tangent and the normal to the hyperbola* $x = 2\sec\theta,\ y = 4\tan\theta$ *at the point where* $\theta = -\frac{\pi}{4}$.

3. *Show that the tangent at the endpoints of a focal chord of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *meet on the corresponding directrix.*

2.6.5 Chord of Contact for the Ellipse and Hyperbola

Chord of Contact for the Ellipse: From any point T external to the ellipse, two tangents can be drawn to the curve, meeting the conic in P and Q. PQ is called the chord of contact of tangents from T.



For ellipse in the figure shown above, $T(x_0,\, y_0)$ lies on both tangents, TP and TQ.

$\therefore\ \frac{x_0x_1}{a^2}+\frac{y_0y_1}{b^2}=1$ and $\frac{x_0x_2}{a^2}+\frac{y_0y_2}{b^2}=1$

Hence both $P(x_1,\, y_1)$ and $Q(x_2,\, y_2)$ satisfy $\frac{x_0x_1}{a^2}+\frac{y_0y_1}{b^2}=1$.

Thus the chord of contact of tangents from $T(x_0,\, y_0)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ have the equation $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

Example 2.6.3 Find the equation of the chord of contact of tangents to the ellipse $3x^2+4y^2=48$ **from the point (6, 4).**

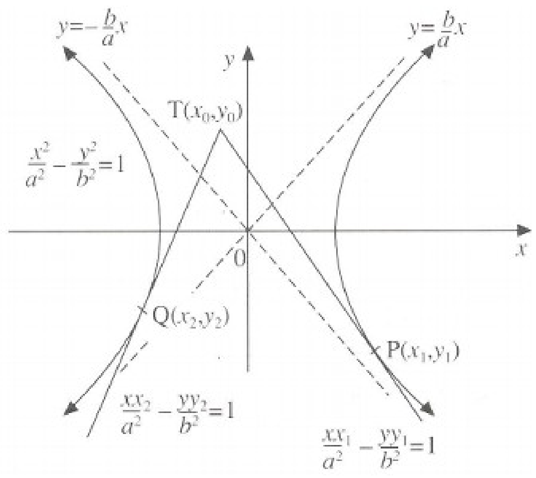
Solution: $3x^2+4y^2=48,\ \Rightarrow\ \frac{x^2}{16}+\frac{x^2}{16}+\frac{y^2}{12}=1.$

The chord of contact of tangents from the point $(6,\, 4)$ *to ellipse* $\frac{x^2}{16}+\frac{y^2}{12}$ *has equation*

$$\frac{6x}{16}+\frac{4y}{12}=1\ \Rightarrow\ 9x+8y=24.$$

Exercise 2.6.2 Find the equation of the chord of contact of tangent from the point (5, 4) to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$**.**

Chord of Contact for the Hyperbola: Similarly $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ is the chord PQ of the hyperbola in the figure shown below.



Thus the chord of contact of tangents from $T(x_0, y_0)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ have the equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

Example 2.6.4 Find the equation of the chord of contact of tangents to the hyperbola $9x^2 - 2y^2 = 18$ from the point $(1, 2)$.

Solution:

$$9x^2 - 2y^2 = 18, \Rightarrow \frac{x^2}{2} - \frac{y^2}{9} = 1.$$

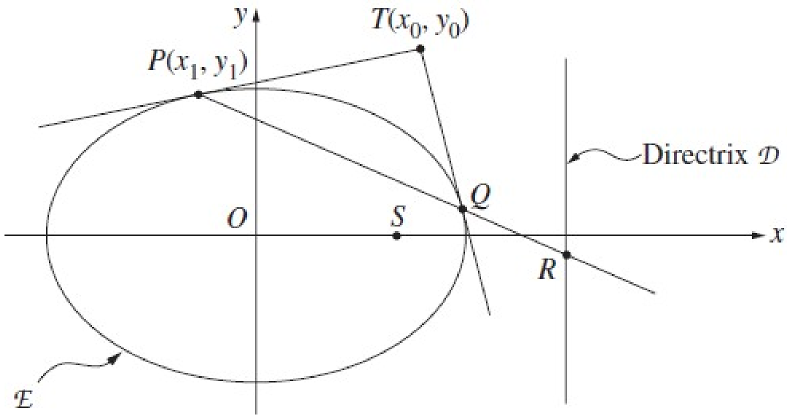
The chord of contact of tangents from the point $(1, 2)$ *the the hyperbola has equation*

$$\frac{x}{2} - \frac{2y}{9} = 1 \Rightarrow 9x - 4y = 18.$$

Exercise 2.6.3 Find the equation of the chord of contact of tangent from the point (1, 2) to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$.

2.7 Pass Exam Questions

Exercise 2.7.1 The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and focus S and directrix D as shown in the diagram. The point $T(x_0, y_0)$ lies outside the ellipse and is not on the x axis. The chord of constant PQ from T intersects D at R, as shown in the diagram.



1. *Show that the equation of the tangent to the ellipse at the point $P(x_1, y_1)$ is $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$.*

2. *Show that the equation of the chord of contact from* $T(x_0,\, y_0)$ *is* $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

3. *Show that* TS *is perpendicular to* SR.

Exercise 2.7.2

1. *Derive the equation of the tangent to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *at the point* $P(a\sec\theta,\ b\tan\theta)$.

2. *Show that the tangent intersects the asymptotes of the hyperbola at the points* $A\left(\frac{a\cos\theta}{1-\sin\theta},\ \frac{b\cos\theta}{1-\sin\theta}\right)$ *and* $B\left(\frac{a\cos\theta}{1+\sin\theta},\ \frac{-b\cos\theta}{1+\sin\theta}\right)$.

3. *Prove that the area of the triangle OAB is ab.*

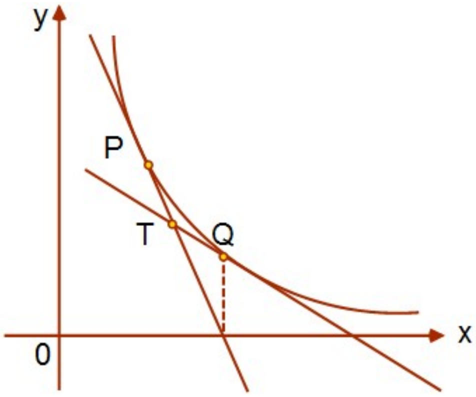
Exercise 2.7.3 Consider the ellipse ϵ with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the point $P(a\cos\theta,\, b\sin\theta)$, $Q(a\cos(\theta+\phi),\, b\sin(\theta+\phi))$ and $R(a\cos(\theta-\phi),\, b\sin(\theta-\phi))$ on ϵ.

1. *Show that the equation of the tangent to ϵ at the point P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.*

2. *Show that the chord QR is parallel to the tangent at P.*

3. *Deduce that OP bisects the chord QR.*

Exercise 2.7.4 The distinct $P(cp,\ \frac{c}{p})$ and $Q(cq,\ \frac{c}{q})$ are on the same branch of the hyperbola H with equation $xy = c^2$. The tangents to H at P and Q meet at the point T.



1. *Show that the equation of the tangent at P is $x + p^2y = 2cp$.*

2. *Show that T is the point $(\frac{2cpq}{p+q},\ \frac{2c}{p+q})$.*

3. *Suppose P and Q move so that the tangent at P intersects the x axis at $(cq,\, 0)$. Show that the locus of T is a hyperbola, and state its eccentricity.*

Exercise 2.7.5 Find the eccentricity , foci and the equation of the directices of the ellpse $\frac{x^2}{9}+\frac{y^2}{4}=1$.