4 Unit Math Homework for Year 12

Student Name: ____________	Grade: ____________
Date: ____________	Score: ____________

Table of contents

This edition was printed on June 16, 2017 with worked solutions.

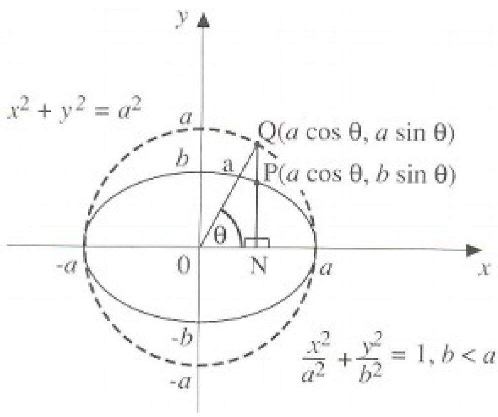
Camera ready copy was prepared with the LaTeX2e typesetting system.

2 Topic 2 — Conics Part 2

2.4 Parametric Equations for the Ellipse and Hyperbola

2.4.1 Parametric Equations for the Ellipse

Definition: Let N be the foot of a perpendicular from a point P on the ellipse to the x-axis.
Produce NP to a point Q on the circle $x^2 + y^2 = a^2$.
Let OQ make an angle θ with the positive x-axis, $\pi < \theta \leq \pi$.
Then Q has coordinates $(a\cos\theta, a\sin\theta)$.
At P, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x = a\cos\theta$, hence $y = b\sin\theta$, where $a > b$.
Hence the ellipse with Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
has parametric equations $x = a\cos\theta$ and $y = b\sin\theta$, where $-\pi < \theta \leq \pi$.
The circle $x^2 + y^2 = a^2$ is called the auxiliary circle.



Exercise 2.4.1 Find the parametric equations of

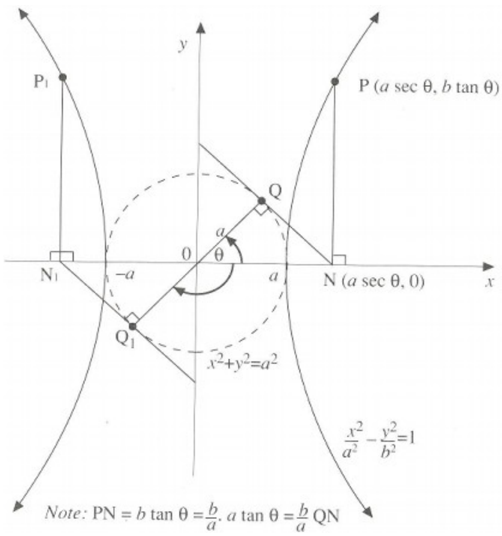
1. *The ellipse* $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

2. *The ellipse* $x^2 + 4y^2 = 4$.

2.4.2 Parametric Equations for Hyperbola

Definition: N the foot of the perpendicular form a point P on the hyperbola to the x-axis.
From N, a tangent is drawn to the circle $x^2 + y^2 = a^2$, meeting the circle in Q.
If P is on the right-hand branch, choose Q in the same quadrant as P.
If P is on the left-hand branch, choose Q in different quadrants as P.
Let θ be the angle OQ makes with the positive x-axis.
At P, $x = a\sec\theta$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, hence $y = b\tan\theta$
For P on the right-hand branch, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
For P on the left-hand branch, $-\pi < \theta < -\frac{\pi}{2}$
The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has parametric equations

$$x = a\sec\theta \quad \text{and} \quad y = b\tan\theta, \text{ where } -\pi < \theta \leq \pi, \theta \neq \pm\frac{\pi}{2}.$$



Exercise 2.4.2 Find the parametric equations of

1. *The hyperbola* $\frac{x^2}{16} - \frac{y^2}{25} = 1$.

2. *The hyperbola* $x^2 - y^2 = 4$.

Exercise 2.4.3 Find the Cartesian equations of:

1. *The ellipse* $x = 3\cos\theta,\ y = 2\sin\theta$.

2. *The ellipse* $x = 5\cos\theta,\ y = 4\sin\theta$.

3. *The hyperbola* $x = 3\sec\theta,\ y = 4\tan\theta$.

4. *The hyperbola* $x = 2\sec\theta,\ y = 5\tan\theta$.

2.4.3 Further Trigonometry

Sums and Differences:

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

Double Angle Identities:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Sums to Products:

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

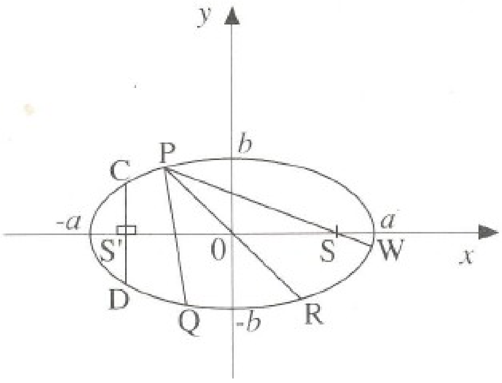
$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

The t-Formulae: Let $t = \tan\frac{\theta}{2}$, Then:

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \tan\theta = \frac{2t}{1-t^2}$$

2.4.4 Equation of a Chord on an Ellipse



PQ, PR, PW are chords of the conic.

Chord PR through the centre is a diameter,

PW through the focus S is a focal chord.

Cd is a focal chord which is perpendicular to the major axis.

Such a chord is termed a latus rectum of the conic and has equation $x = \pm ae$.

The equation of a general chord PQ is best expressed in terms of the parameters θ and ϕ of the end points P and Q.

Definition: Let $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the gradient of PQ is:

$$\frac{b(\sin\theta - \sin\phi)}{a(\cos\theta - \cos\phi)} = \frac{b}{a} \times \frac{2\sin(\frac{\theta-\phi}{2})\cos(\theta+\phi)}{-2\sin(\frac{\theta-\phi}{2})\sin(\frac{\theta+\phi}{2})} = -\frac{b}{a} \times \frac{\cos(\frac{\theta+\phi}{2})}{\sin(\frac{\theta+\phi}{2})}$$

$$\therefore y - b\sin\theta = -\frac{b}{a} \times \frac{\cos(\frac{\theta+\phi}{2})}{\sin(\frac{\theta+\phi}{2})}(x - a\cos\theta)$$

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\theta\cos\left(\frac{\theta+\phi}{2}\right) + \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)$$

Then the equation of the chord is: $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos(\frac{\theta-\phi}{2})$.

In the special case where PQ is a diameter, $(0, 0)$ lies on the chord and hence $|\theta - \phi| = \pi$

Example 2.4.1 $P(a\cos\theta,\ b\sin\theta)$ **and** $Q[a\cos(\pi+\theta),\ b\sin(\pi+\theta)]$ **lie on the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**. Show that** PQ **passes through** $(0,\ 0)$**.**

Solution: *The equation of the chord PQ of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is:*

$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$, *where P, Q have parameters* $\theta,\ \phi$

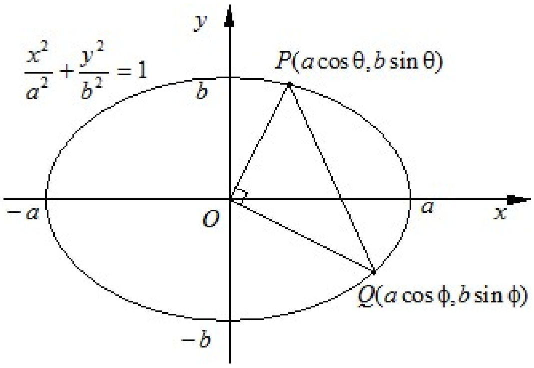
We have $\phi = \pi + \theta$. *Hence the equation of the chord PQ transforms into:*

$\frac{x}{a}\cos\left(\frac{\theta+(\pi+\theta)}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+(\pi+\theta)}{2}\right) = \cos\left(\frac{\theta-(\pi+\theta)}{2}\right)$.

$\frac{x}{a}\cos\left(\frac{2\theta+\pi}{2}\right) + \frac{y}{b}\sin\left(\frac{2\theta+\pi}{2}\right) = \cos\left(\frac{-\pi}{2}\right)$.

Thus $-\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 0$. *Therefore (0, 0) lies on the chord PQ.*

Exercise 2.4.4 The point $P(a\cos\theta,\ b\sin\theta)$, and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,\ 0)$. Show that $\tan\theta\tan\phi=-\frac{a^2}{b^2}$.

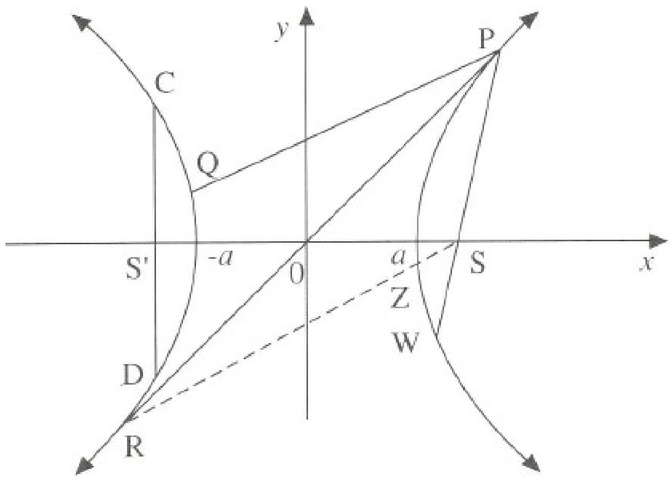


Exercise 2.4.5 The point $P(a\cos\theta,\ b\sin\theta)$ and $Q[a\cos(-\theta), b\sin(-\theta)]$ are the extremities of the latus
rectum $x=ae$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

1. *Show that* $\cos\theta=e$.

2. *Show that PQ has length* $\frac{2b^2}{a}$.

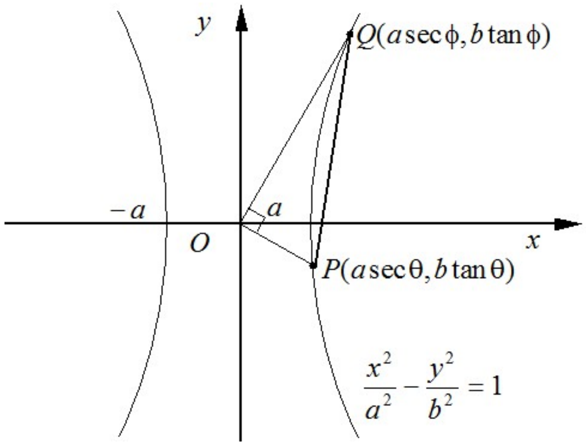
2.4.5 Equation of a Chord on Hyperbola



Definition: Let $P(a\sec\theta,\ b\tan\theta)$ and $Q(a\sec\phi,\ b\tan\phi)$ lie on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
Then the equation of the chord is: $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos(\frac{\theta+\phi}{2})$.

In the special case where PQ is a diameter, $(0,0)$ lies on the chord and hence $|\theta+\phi|=\pi$

Example 2.4.2 The points $P(a\sec\theta,\ b\tan\theta)$ **and** $Q(a\sec\phi,\ b\tan\phi)$ **lie on the hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **and the chord PQ subtends a right angle at** $(0,\ 0)$**. Show that** $\sin\theta\sin\phi=-\frac{a^2}{b^2}$**.**



Solution: *POQ is a right-angled triangle. Therefore* $OP^2+OQ^2=PQ^2$.

$$\begin{aligned}a^2\sec^2\theta+b^2\tan^2\theta+a^2\sec^2\phi+b^2\tan^2\phi &= a^2(\sec\theta-\sec\phi)^2+b^2(\tan\theta-\tan\phi)^2\\ &= a^2(\sec^2\theta-2\sec\theta\sec\phi+\sec^2\phi)\\ &+b^2(\tan^2\theta-2\tan\theta\tan\phi+\tan^2\phi)\end{aligned}$$

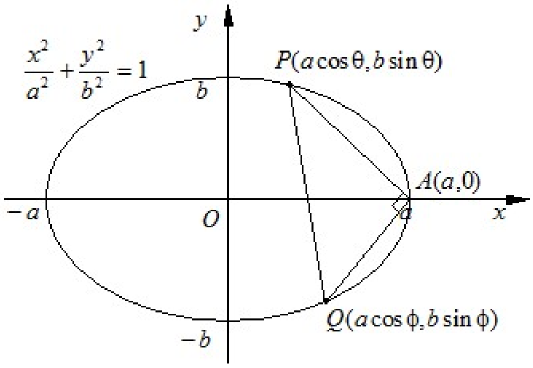
Then $0=-2a^2\sec\theta\sec\phi-2b^2\tan\theta\tan\phi,\ \Rightarrow\therefore\sin\theta\sin\phi=-\dfrac{a^2}{b^2}$

Exercise 2.4.6 $P(a\sec\theta,\, b\tan\theta)$ **lie on the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **with foci** $S(ae,\, 0)$ **and** $S'(-ae,\, 0)$**.**

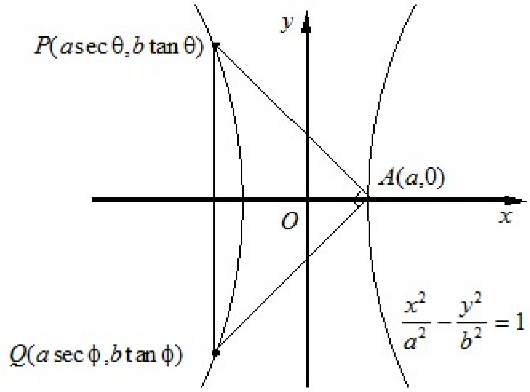
1. *Show that* $PS = a(e\sec\theta - 1)$ *and* $PS' = a(e\sec\theta = 1)$

2. *Deduce that* $|PS - PS'| = 2a$*.*

Exercise 2.4.7 The point $P(a\cos\theta,\ b\sin\theta)$ **and** $Q(a\cos\phi,\ b\sin\phi)$ **lie on the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**. If** PQ **subtends a right angle at** $(a,\ 0)$ **show that** $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$**.**



Exercise 2.4.8 The points $P(a\sec\theta,\, b\tan\theta)$ **and** $Q(a\sec\phi,\, b\tan\phi)$ **lie on the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **If PQ subtends a right angle at** $(a,\, 0)$**. Show that** $\tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{b^2}{a^2}$.



Exercise 2.4.9

1. *Given that the equation of the chord PQ of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* $\frac{x}{a}\cos(\frac{\theta+\phi}{2})+\frac{y}{b}\sin(\frac{\theta+\phi}{2})=\cos(\frac{\theta-\phi}{2})$, *where P, Q have parameters* $\theta,\ \phi$, *show that if PQ is a focal chord then* $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ *takes one of values* $-\frac{1-e}{1+e}$ *or* $\frac{1+e}{1-e}$.

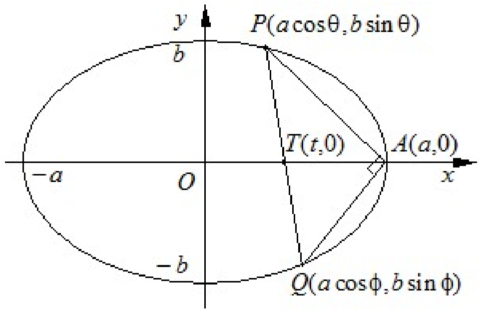
2. *PQ is a focal chord of* $\frac{x^2}{4}+\frac{y^2}{3}=1$, *where P has coordinates* $(1,\ \frac{3}{2})$. *Find the coordinate of Q.*

Exercise 2.4.10

1. $P(a\sec\theta,\, b\tan\theta)$ *and* $Q(a\sec\phi,\, b\tan\phi)$*, lie on the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$*. Use the result that the chord PQ has equation* $\frac{x}{a}\cos(\frac{\theta-\phi}{2}) - \frac{y}{b}\sin(\frac{\theta+\phi}{2}) = \cos(\frac{\theta+\phi}{2})$ *to show that if PQ is a focal chord, then* $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ *takes one of the values* $\frac{1-e}{1+e}$ *or* $\frac{1+e}{1-e}$.

2. $P(2\sqrt{3},\, 3\sqrt{3})$ *is one extremity of a focal chord on the hyperbola* $\frac{x^2}{3} - \frac{y^2}{9} = 1$*. Find the coordinates of the other extremity Q.*

Exercise 2.4.11 Points $P(a\cos\theta,\, b\sin\theta$ **and** $Q(a\cos\phi,\, bsin\phi)$ **lie on the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**. Find the equation of the chord PQ. Hence show that if PQ subtends a right angle at the point** $A(a,\,0)$ **then PQ passes through a fixed point on the x-axis.**



2.5 Past Exam Questions