# 4 Unit Math Homework for Year 12

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| **Student Name**: ____________ | **Grade**: ____________ |
| **Date**: ____________ | **Score**: ____________ |

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This edition was printed on June 16, 2017 with worked solutions.

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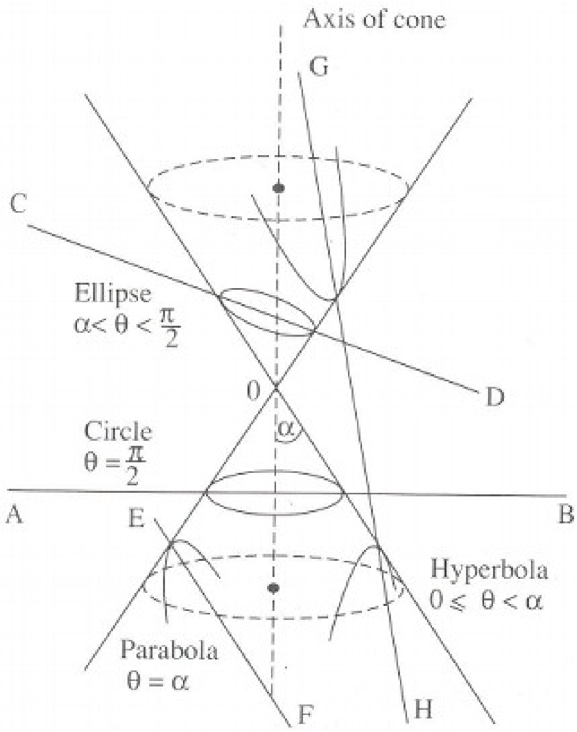
# 2 Topic 2 — Conics Part 1

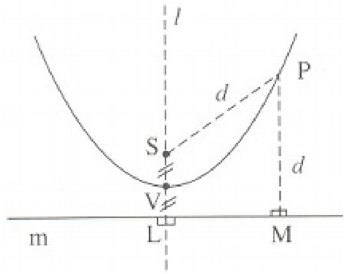
## 2.1 Focus/dirctrix Definitions and Cartesian equations

**Definition:** The Curve formed when a plane cuts a double cone is called a conic section. The nature of curve is determined by the relation between the semi-vertical angle $\alpha$ of the cone and the angle $\theta$ between the plane and axis of the cone.

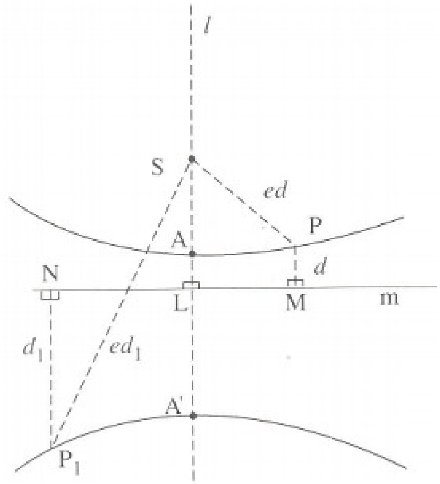


**Definition:** Let S be a fixed point called the focus of the conic and m a fixed line called the directrix in a plane.
Define the locus of a point P in the plane by the condition that the ratio of the distance from P to S and from P to M is $e : 1$ for some constant $e > 0$.
The shape of the curve is determined by the value of e, which is called the eccentricity of the curve.

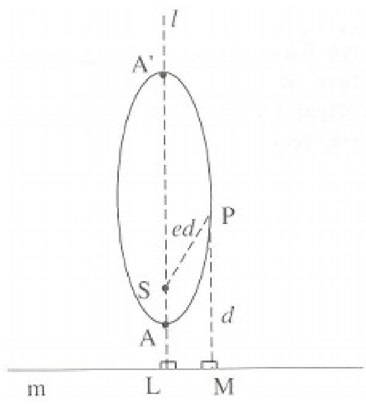
Case 1: Parabola when $e = 1$ $(e = PS : PM = d : d = 1)$.




Case 2: Hyperbola when $e > 1$ $(e = PS : PM = ed : d > 1)$.




Case 3: Ellipse when $0 < e < 1$ $(e = PS : PM = ed : d < 1)$



### 2.1.1 Cartesian Equation of the Ellipse

**Definition:** Let A, S, L have the coordinates (a, 0), (k, 0), (c, 0) respectively, where $a > 0$

The focus S has coordinates $(ae,\ 0)$ and the directrix m has equation $x = \frac{a}{e}$.

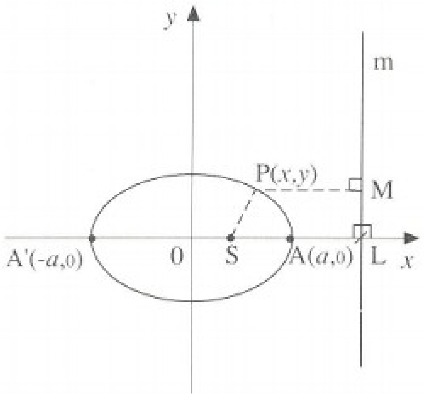
Let $P(x,\ y)$ lie on the curve. Then M has coordinates $(\frac{a}{e},\ y)$.

For the ellipse, $0 < e < 1$ and $PS = e \times PM$,

$\Rightarrow\ (x - ae)^2 + y^2 = e^2(x - \frac{a}{e})^2\ \Rightarrow\ x^2(1 - e^2) + y^2 = a^2(1 - e^2)$,

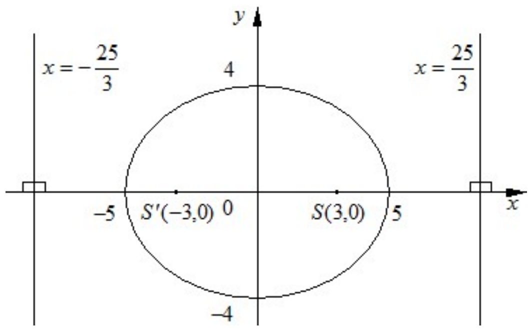
We let $b^2 = a^2(1 - e^2)$, where $b > 0$

The Cartesian equation of the ellipse is then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.




**Example 2.1.1 For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices. Sketch the ellipse.**

***Solution:***




$$\frac{x^2}{25} + \frac{y^2}{16} = 1,\ \Rightarrow\ \frac{x^2}{5^2} + \frac{y^2}{4^2},\ \Rightarrow\ a = 5,\ b = 4\ \textit{and}\ a > b,$$

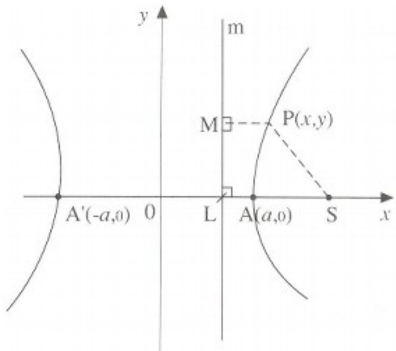
$$b^2 = a^2(1 - e^2),\ \ \textit{eccentricity:}\ e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}.$$

$$\textit{foci:}\ (ae,\ 0)\ \Rightarrow\ (\pm 3,\ 0).$$

$$\textit{directrices:}\ x = \pm\frac{a}{e}\ \Rightarrow\ x = \pm\frac{5}{\frac{3}{5}} = \pm\frac{25}{3}.$$

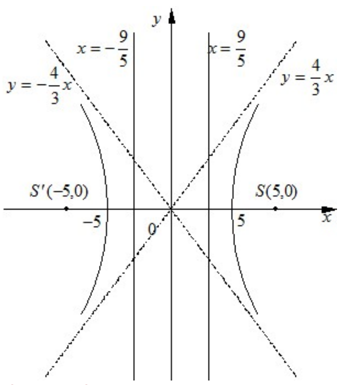
### 2.1.2 Cartesian Equation of the Hyperbola

**Definition:** Let A, S, L have the coordinates (a, 0), (k, 0), (c, 0) respectively, where $a > 0$
The focus S has coordinates $(ae,\ 0)$ and the directrix m has equation $x = \frac{a}{e}$.
Let $P(x,\ y)$ lie on the curve. Then M has coordinates $(\frac{a}{e},\ y)$.
For the hyperbola, $e > 1$ and $PS = e \times PM$,
$\Rightarrow\ (x - ae)^2 + y^2 = e^2(x - \frac{a}{e})^2 \ \Rightarrow\ x^2(1 - e^2) + y^2 = a^2(1 - e^2)$,
We let $b^2 = a^2(e^2 - 1)$, where $b > 0$
The Cartesian equation of the hyperbola is then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.




**Example 2.1.2 For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find (a) the eccentricity, (b) the coordinates of the foci, (c) the equation of the directrices, (d) the equation of the asymptotes. Sketch the hyperbola.**

***Solution:***




$$\frac{x^2}{9} - \frac{y^2}{16} = 1,\ \Rightarrow\ \frac{x^2}{3^2} - \frac{y^2}{4^2} = 1,\ \Rightarrow\ a = 3,\ \textit{and}\ b = 4.$$

$$b^2 = a^2(e^2 - 1),\ \textit{eccentricity: } e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}.$$

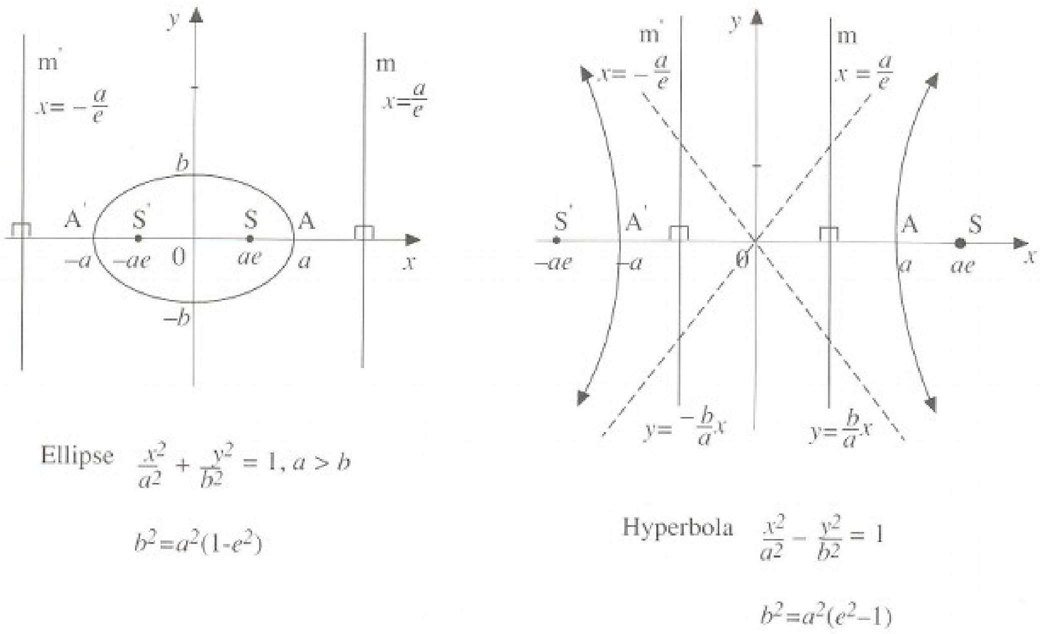
$$\textit{foci: } (\pm ea,\ 0)\ \Rightarrow\ (\pm 5,\ 0).$$

$$\textit{directrices: } x = \pm\frac{a}{e}\ \Rightarrow\ x = \pm\frac{3}{\frac{5}{3}} = \pm\frac{9}{5}.$$

$$\textit{asymptotes: } y = \pm\frac{b}{a}x,\ \Rightarrow\ y = \pm\frac{4}{3}x.$$

### 2.1.3 Foci of the Ellipse and Hyperbola

**Definition:** The ellipse and hyperbola has two foci $S(ae,\, o)$ and $S'(-ae,\, 0)$
and two directrices $m:\; x = \frac{a}{e}$ and $m':\; x = -\frac{a}{e}$
Thus the length of the major axis $AA'$ is $2a$, the length of the minor axis is $2b$.
For Ellipse: $b^2 = a^2(1-e^2)$, for Hyperbola: $b^2 = a^2(e^2-1)$.
The hyperbola has two asymptotes: $y = \pm\frac{b}{a}x$.




**Exercise 2.1.1 Fore the ellipse $\frac{x^2}{3} + \frac{y^2}{2} = 1$ find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices. Sketch the ellipse.**

**Exercise 2.1.2**

1. *For the ellipse* $x^2 + 2y^2 = 4$, *find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices. Sketch the ellipse.*

2. *For the hyperbola* $\frac{x^2}{2} - \frac{y^2}{4} = 1$, *find (a) the eccentricity, (b) the coordinates of the foci, (d) the equations of the directrices, (d) the equation of the asymptotes. Sketch the hyperbola.*

**Exercise 2.1.3**

1. *For the hyperbola $x^2 - y^2 = 4$, find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices, (d) the equations of the asymptotes. Sketch the hyperbola.*

2. *The ellipse has eccentricity $\frac{2}{3}$ and directrices $x = -9$ and $x = 9$. Find the equation of the ellipse.*

**Exercise 2.1.4**

1. *The ellipse has eccentricity* $\frac{4}{5}$ *and foci (4, 0) and (-4, 0). Find the equation of this ellipse.*

2. *The hyperbola has eccentricity* $\frac{3}{2}$ *and directrices* $x = -4$ *and* $x = -4$*. Find the equation of this hyperbola.*

3. *The ellipse has eccentricity* $\frac{2}{3}$ *and directrices* $x = -9$ *and* $x = 9$*. Find the equation of the ellipse.*

**Exercise 2.1.5**

1. *A variable point $P(x, y)$ moves so that its distance from $(0, 1)$ is one-half its distance from $y = 4$. Find the locus of $P$.*

2. *An ellipse has its centre at the origin and its foci on the x-axis. The distance between the foci is 4 units and the distance between the directrices is 16 units. Find the equation of ellipse.*