4 Unit Math Homework for Year 12

Student Name: ________________	**Grade**: ________________
Date: ________________	**Score**: ________________

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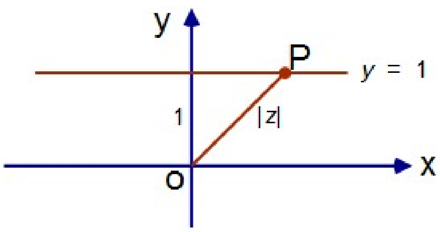
1 Topic 1 — Complex Numbers Part 4

1.12 Curves and regions in the Argand diagram

Definition: Let P be the representation on an Argand diagram of a complex number z which satisfies the equation $\Re z = 3$.
Let $z = x + iy$. Then $Re\, z = 3 \;\Rightarrow\; x = 3 \Rightarrow P$ lies in the line $x = 3$.
$\therefore$ the equation $\Re z = 3$ defines the line $x = 3$ in the Argand diagram.

Example 1.12.1 Sketch the curve in the Argand diagram defined by the equation $Im(z-1+3i) = 4$. Find the minimum value of |z| subject to this condition and state the value of z of z which this minimum is attained.

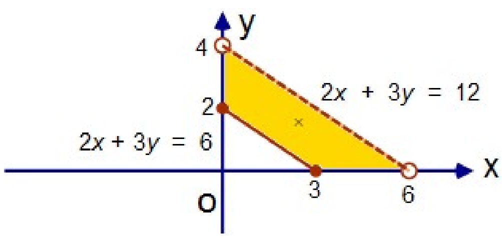
Solution:



Let $z = x = iy$. *then* $z = 1 + 3i = (x-1) + i(y+3)$,
$\therefore Im(z-1+3i) = 4, \;\Rightarrow\; y + 3 = 4, \;\Rightarrow\; y = 1.$
The curve has Cartesian equation $y = 1$.

Example 1.12.2 Sketch the region in the Argand diagram define by $6 \le Re\,[(2-3i)z] < 12$ and $Re\, z.Im z \ge 0$.

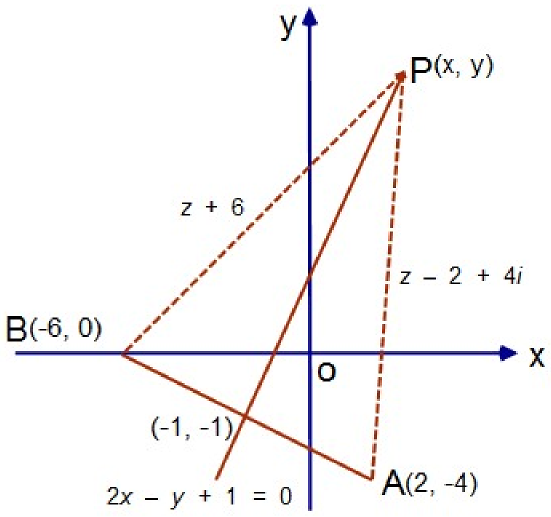
Solution:



Let $z = x + iy$. *Then* $(2-3i)(x+iy) = (2x+3y) + i(2y-3x)$.
$6 \le Re\,[(2-3)i)z < 12 \;\Rightarrow\; 6 \le 2x + 3y < 12,$
and $Re\, z.Im\, z \ge 0 \;\Rightarrow\; xy \ge 0.$

Example 1.12.3 z satisfies $|z-2+4i| = |z+6|$. Sketch the locus of the point P representing z in the Argand diagram ad find its Cartesian equation. Hence find the minimum value of |z|.

Solution:



Let P and A and B represent z, $2-4i$ and -6 respectively.
Then $\overrightarrow{AP}$, $\overrightarrow{BP}$ *represent* $z-(2-4i)$, $z-(-6)$,
and $|z-2+4i| = |z+6| \Rightarrow, AP = BP$.
The locus of P is the perpendicular bisector of AB.
Since AB has midpoint $(-1, -1)$ and gradient $-\frac{1}{2}$,
the locus of P passes through $(-1,-1)$ *with gradient* 2
and has Cartesian equation $2x-y+1=0$.
Now $OP = |z|$. Hence the minimum value of |z| is perpendicular distance from $(0, 0)$ *to the locus of P.*
Therefore the minimum value of $|z|$ is $\frac{1}{\sqrt{5}}$.

Exercise 1.12.1 Sketch the locus specified by:

1. $|z| \leq |z-2|$ *and* $-\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}$.

2. $Re(z - \frac{1}{z}) = 0$, *(state the equation(s) of the locus).*

3. *Sketch on an Argand diagram the locus of the point P representing z, given that* $|z|^2 = z + \overline{z} + 1$.

1.13 Past HSC Exam Questions

Exercise 1.13.1 Let $z = 3 + i$ and $w = 1 - i$. Find in the form $x + iy$,

1. $2z + iw$

2. $\overline{z}w$

3. $\frac{6}{w}$.

Exercise 1.13.2 Let $\beta = 1 - i\sqrt{3}$

1. *Express β in modulus-argument form.*

2. *Express β^7 in modulus-argument form.*

3. *Hence express β^7 in the form $x + iy$.*

Exercise 1.13.3 Sketch the region on the Argand diagram where the inequalities $|z - \overline{z}| < 2$ and $|z - 1| \geq 1$ hold simultaneously.

Exercise 1.13.4 Let ℓ be the line in the complex plane that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$. The point P represents the complex number z_1 where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line ℓ to produce the point Q, where represents the complex number z_2. Hence $|z_1| = |z_2|$.

1. *Explain why* $\arg(z_1) + \arg(z_2) = 2\alpha$.

2. *Deduce the* $z_1 z_2 = |z_1|^2(\cos 2\alpha + i \sin 2\alpha)$.

3. *Let* $\alpha = \frac{\pi}{4}$ *and let R be the point that represents the complex number* $z_1 z_2$. *Describe the locus of R as* z_1 *varies.*

Exercise 1.13.5

1. *Find the modulus and principle argument of* z *if* $z = 1 + i\sqrt{3}$*. Hence find the smallest positive integer* n *such that* z^n *is a real number.*

2. *Find both complex roots of* $\omega^2 = 3 + 4i$*. Hence find the complex roots of the equation* $z^2 + (4 - 2i)z - 8i = 0$.

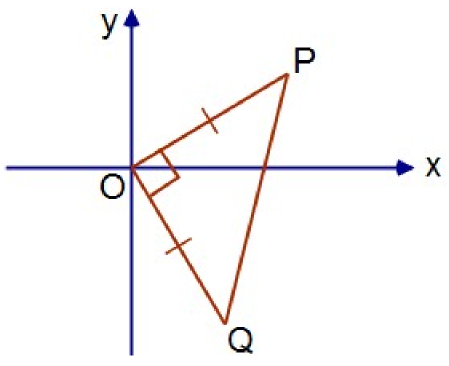
3. *Find all the complex numbers that satisfy* $|z|^2 - iz = 36 + 4i$.

Exercise 1.13.6

1. *Evaluate* $\arg[(2+i)\overline{\omega}]$, *given that* $\omega = -1 - 3i$.

2. *Write* $x^2 - 12x + 48$ *as the product of two linear factors.*

3. *In the diagram given below, triangle* POQ *is an isosceles right-angled triangle. If* P *represents the complex number* $a + bi$, *where* a *and* b *are real, find the complex number represented by* Q.



Exercise 1.13.7

1. *Find, in modulus-argument form, the three cube roots of -8.*

2. *Write the two unreal cube roots of -8 in the form a + bi, where a and b are real.*

3. *If* ω_1 *and* ω_2 *are the unreal cube roots of -8, show that* $\omega^{6n} + \omega^{6n} = 2^{6n+1}$ *for all integers* n*.*

Exercise 1.13.8

1. *Express* $\sqrt{3}+i$ *and* $\sqrt{3}-i$ *in modulus -argument form.*

2. *Hence, simplify* $(\sqrt{3}+i)^{15}+(\sqrt{3}-i)^{15}$.

3. *The points A and C represent the complex numbers* $z_1=1+i$ *and* $z_3=7+3i$. *Find the complex number* z_2 *represented by B such that* $\triangle ABC$ *is isosceles and right angled at* B.