4 Unit Math Homework for Year 12

Student Name: ______________	Grade: ______________
Date: ______________	Score: ______________

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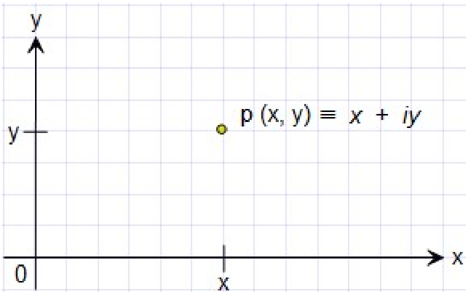
1 Topic 1 — Complex Numbers Part 2

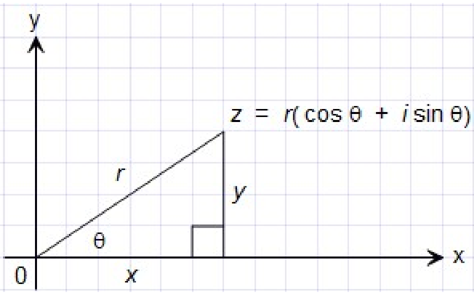
1.4 Geometrical Representation

1.4.1 Modulus and argument of a complex number

Definition: For each complex number $z = x + iy$, we may plot a corresponding point $P(x, y)$ in the Cartesian Plant.
The x-axes and y-axes are refereed as the real and imaginary axes respectively.

$$P(x,\, y) = x + iy$$





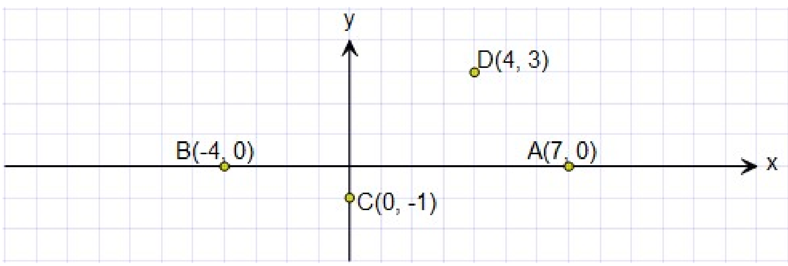
Definition: Let $z = x + iy$ be complex number: then:
The polar form of the complex number is: $z = r(\cos\theta + i\sin\theta)$.
The length $r = |z| = \sqrt{x^2 + y^2}$ is called the **modulus** of z, and
The angle $\theta = arg\, z$ is called the **principle argument** of z where: $-\pi < \theta \leq \pi$

$$\theta = \begin{cases} \tan^{-1}(\frac{y}{x}), & \text{if } z \text{ is in 1st or 4th quadrants,} \\ \tan^{-1}(\frac{y}{x}) + \pi, & \text{if } z \text{ is 2nd quadrant,} \\ \tan^{-1}(\frac{y}{x}) - \pi, & \text{if } z \text{ is in 3rd quadrant.} \end{cases}$$

An important consequent result is: $r^2 = |z|^2 = z\bar{z}$

Example 1.4.1 On an Argand diagram show the point $A,\ B,\ C,\ D,$ **representing the complex numbers** $7,\ -4,\ -i,\ 4 + 3i$ **respectively.**

Solution:



Example 1.4.2 Find the modulus and principal argument of:

1. $z = 3 + 5i$

Solution: $z = 3 + 5i, \Rightarrow |z| = \sqrt{3^2 + 5^2} = \sqrt{34}$
$\arg z = \tan^{-1} \frac{5}{3}$.

2. $z = -2 + 3i$

Solution: $z = -2 + 3i, \Rightarrow |z| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$,
$\arg z = \tan^{-1} \frac{3}{-2} + \pi = \pi - \tan^{-1} \frac{3}{2}$.

3. $z = 2 - 4i$

Solution: $z = 2 - 4i, \Rightarrow |z| = \sqrt{2^2 + (-4)^2} = 2\sqrt{5}$
$\arg z = \tan^{-1} \frac{-4}{2} = -\tan^{-1} 2$.

Exercise 1.4.1 Find the modulus and principal argument of:

1. $3 - 3i$

2. $-2 + 4i$

3. $-2i$

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Example 1.4.3 Write $-2-4i$ in modulus-argument form.

Solution: *Let* $z=-2-4i,\ \Rightarrow |z|=\sqrt{4+16}=2\sqrt{5}$

$\theta=-\pi+\tan^{-1}\frac{4}{2},\ \Rightarrow\ \arg z=-\pi+\tan^{-1}2.$

$\therefore z=2\sqrt{5}\left[\cos(-\pi+\tan^{-1}2)+i\sin(-\pi+\tan^{-1}2)\right].$

To simplify the appearance of such expressions, $\cos\theta+i\sin\theta$

is often abbreviated to ***cis*** θ.

for example $z=2\sqrt{5}\,cis\,(-\pi+\tan^{-1}2)$

Exercise 1.4.2 Express the following in modulus-argument form:

1. $-2\sqrt{3}-2i$

2. $\sqrt{3}-i$

3. $-1+\sqrt{3}i$

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Exercise 1.4.3 If k is a positive real number, write z in modulus-argument form in terms of k if:

1. $z = k$

2. $z = -k$

3. $z = ki$

4. $z = -ki$

Exercise 1.4.4 Express $-2\sqrt{3} + 2i$ in modulus-argument form.

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1.4.2 Products and quotients using modulus-argument form

Definition: Let $z_1 = r_1(\cos\alpha + i\sin\alpha),\ z_2 = r_2(\cos\beta + i\sin\beta)$,
where $r_1 = |z_1|$ and $r_2 = |z_2|$ are positive real numbers and $\arg z_1 = \alpha$, $\arg z_2 = \beta$

$$\begin{aligned} \text{Then } z_1z_2 &= r_1r_2(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= r_1r_2[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)] \\ &= r_1r_2[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] \end{aligned}$$

$\therefore |z_1z_2| = r_1r_2 = |z_1||z_2|$ and $\alpha + \beta$ is one value of $\arg(z_1z_2)$

Example 1.4.4 If $z_1 = 3(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ **and** $z_2 = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$**, write** z_1z_2**:**

1. *in modulus-argument form,*

Solution: $|z_1| = 3,$ *and* $|z_2| = 2,\ \Rightarrow\ |z_1z_2| = 3\times 2 = 6,$

$\arg z_1 = \dfrac{2\pi}{3}$ *and* $\arg z_2 = \dfrac{5\pi}{6},\ \Rightarrow\ \arg z_1z_2 = \dfrac{2\pi}{3} + \dfrac{5\pi}{6} = \dfrac{3\pi}{2}$

But $\dfrac{3\pi}{2} > \pi$. *The principle argument of* $z_1z_2 = \dfrac{3\pi}{2} - 2\pi = -\dfrac{\pi}{2}$

$\therefore z_1z_2 = 6\left[\cos(-\dfrac{\pi}{2}) + i\sin(-\dfrac{\pi}{2})\right].$

2. *in the form* $a + ib$

Solution: $z_1z_2 = 6(0 - i) = -6i.$

Exercise 1.4.5 If $z_1 = 1 - i$ **and** $z_2 = -1 + \sqrt{3}i$

1. *Find the moduli and principal arguments of* $z_1,\ z_2$ *and* z_1z_2.

2. *Use the given forms of* z_1, z_2 *to find the form* $a + ib$. *Hence evaluate* $\cos\frac{5\pi}{12}$ *as a surd.*

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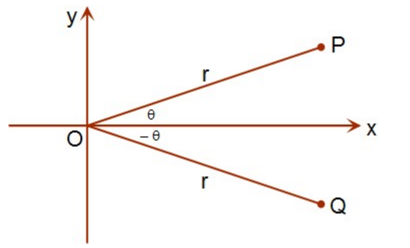
1.4.3 Geometrical relationship between points on an argument diagram

Definition: P, Q represent $z, \bar{z}$ respectively, where $z = r(\cos\theta + i\sin\theta)$

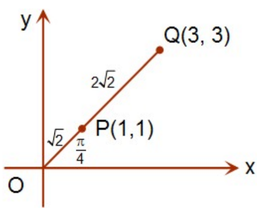


P and Q are reflection of each other in the real axis.

Example 1.4.5 Describe the following transformations and illustrate on an Argand diagram for $z = 1 + i$.

1. $z \to 3z$

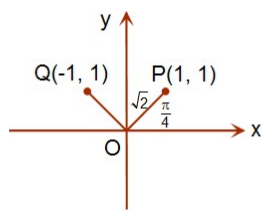
Solution:



Let P, Q represent $z, 3z$ on an Argand diagram.
$|3z| = 3|z| \;\Rightarrow\; OQ = 3OP$
$\arg(3z) = 0 + \arg z = \arg z \;\Rightarrow$ *P, O lie on the same ray from O,*
$\therefore$ transformation $z \to 3z$ is an enlargement about O by a factor 3.
Let $z = 1 + i$, Then $P(1,\ 1)$, $Q(3,\ 3)$ represent z, $3z$.

2. $z \to iz$

Solution:



Let P, Q represent z, iz on an Argand diagram.
$|iz| = |i||z| = |z| \;\Rightarrow\; OQ = OP,\;\; \arg(z) = \frac{\pi}{4},$
$\arg(iz) = \frac{\pi}{2} + \arg(z) = \frac{3\pi}{4} \;\Rightarrow$ *ray OQ makes an angle $\frac{\pi}{2}$ with ray OP.*
The transformation $z \to iz$ is a rotation anticlockwise about O thought $\frac{\pi}{2}$.
Let $z = 1 + i$. then $P(1, 1)$, $Q(-1,\ 1)$ represent z, iz.
$\therefore\;\; |iz| = |z| = \sqrt{2}$ *and* $\arg(iz) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$.

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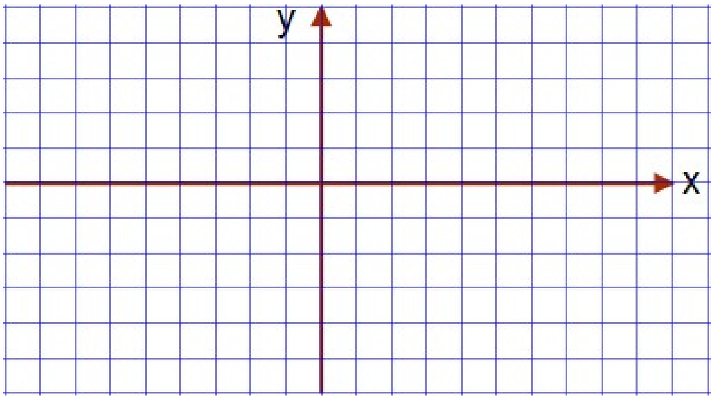
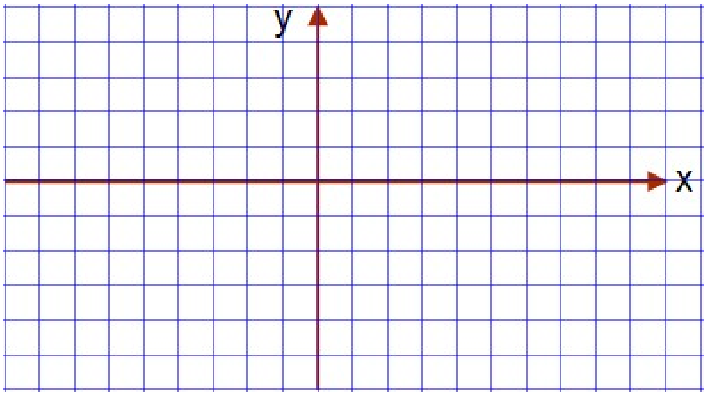
Exercise 1.4.6

1. *Describe geometrically the transformation* $z \to -2z$, *Illustrate on an Argand diagram for* $z = 1 + i$.



2. *Describe geometrically the transformation* $z \to \alpha z$, *where* $\alpha = -2 - 2i$.. *Illustrate on an Argand diagram for* $z = 3i$.



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1.5 Practical Exam Questions

Exercise 1.5.1 Use the properties of modulus and argument of a complex number to deduce the following:

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

2. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

3. $\overline{z_1 z_2} = \overline{z_1} \times \overline{z_2}$

4. $\overline{z_1 \div z_2} = \overline{z_1} \div \overline{z_2}$

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Definition:

$$|z_1 z_2| = |z_1||z_2| \text{ and } \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$
$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \text{ and } \arg\left(\frac{1}{z}\right) = -\arg z$$
$$|z^n| = |z|^n \text{ and } \arg(z^n) = n.\arg(z).$$

Exercise 1.5.2 $z_1 = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ **and** $z_2 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. **Write down the modulus and principal argument of:**

1. z_1^3

2. $\frac{1}{z_2}$

3. $z_1^2 z_2^2$

4. $\frac{z_1^3}{z_2}$

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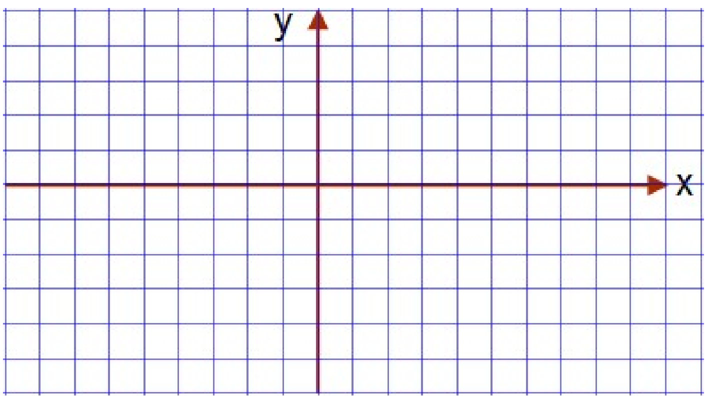
1.6 Past HSC Exam Questions

Exercise 1.6.1 Let $z = \frac{-i}{1+i\sqrt{3}}$

1. *Find the modulus and argument of z.*

2. *Plot Z on the argand diagram.*



Exercise 1.6.2 Let $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$.

1. *Find $\frac{z_1}{z_2}$ in the form of $x + iy$.*

2. *Express z_1 in modulus-argument form.*

3. *Given that z_2 has the modulus-argument form $z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, find the modulus-argument form of $\frac{z_1}{z_2}$*

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4. *Hence find the exact value of* $\sin \frac{\pi}{12}$.

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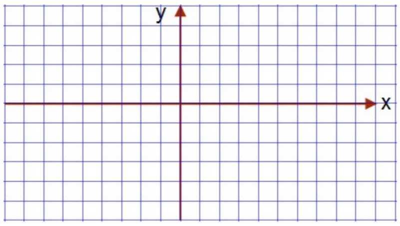
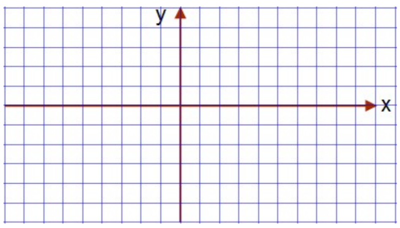
Exercise 1.6.3 Let $A = 1 + 2i$, $B = -3 + 4i$ and $Z = x + iy$. Draw clearly labelled sketches to show the loci satisfied on the Argand diagram by:

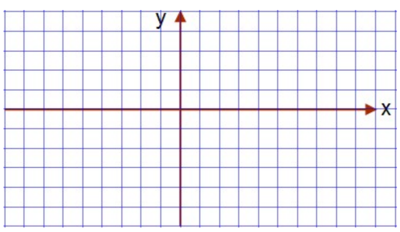
1. $|Z - A| = |B|$



2. $|Z - A| = |Z - B|$



3. $\arg(Z - A) = \frac{\pi}{4}$



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1.7 Miscellaneous Exercise

Exercise 1.7.1 Find the modulus and principal arguments of:

1. $3 + 2i$

2. $-2 + 3i$

3. $-1 - \sqrt{3}i$

4. $i(i - 1)$

5. $\sqrt{3} - i$

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Exercise 1.7.2 Write down the moduli and arguments of $-\sqrt{3}+i$ and $4+4i$. Hence express in modulus-argument form:

1. $(-\sqrt{3}+i)(4+4i)$

2. *If $z = 1+\sqrt{3}i$. Find the smallest positive integer n for which z^n is real and evaluate z^n for this value of n. Show that there is no integral value of n for which z^n is imaginary.*

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Exercise 1.7.3 **z has modulus r and argument θ. Find in terms of r and θ the modulus and one argument of**

1. z^2

2. $\frac{1}{z}$

3. iz

Exercise 1.7.4 **Use the method of mathematical induction to prove that $|z^n| = |z|^n$ and $\arg(z^n) = n \arg z$ for all positive integers n.**

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Example 1.7.1 State the modulus and argument of $-1+i$. Hence write $(-1+i)^{18}$ in the form of $a+ib$.

Solution: *Let* $z=-1+i$. *Then* $z=\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

So $|z|=\sqrt{2}$ *and* $\arg(z)=\frac{3\pi}{4}$.

Hence $|z^{18}|=|z|^{18}=2^9=512$,

and $\arg(z^{18})=18\arg(z)=18\times\frac{3\pi}{4}=\frac{27\pi}{2}=13\frac{\pi}{2}$. *or* $14\pi-\frac{\pi}{2}$.

$\therefore z^{18}=512\left(\cos(14\pi-\frac{\pi}{2})+i\sin(14\pi-\frac{\pi}{2})\right)=512\times(-i)=-512i.$

$\therefore |-1+i|=\sqrt{2},\ arg(-1+i)=\frac{3\pi}{4},\ \Rightarrow\ (-1+i)^{18}=-512i.$

Exercise 1.7.5

1. *Write* $(1+\sqrt{3}i)^{-1}$ *in modulus-argument form.*

2. *Write* $\sqrt{3}+i$ *and* $\sqrt{3}-i$ *in modulus-argument form. Hence write* $\sqrt{3}+i)^{10}+(\sqrt{3}-i)^{10}$ *in the form of* $a+ib$.

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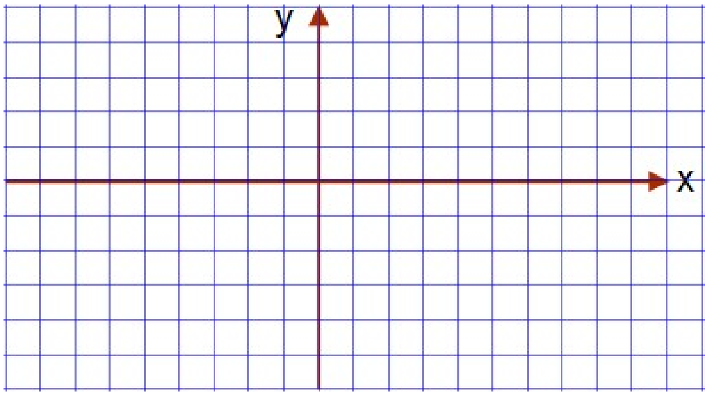
Exercise 1.7.6 *

1. *Obtain in the form* $a+ib$ *the roots of the equation* $x^2+2x+3=0$ *Find the modulus and argument of each root and represent the roots on an Argand diagram by the points A and B.*



2. *Find the modulus of* $\frac{7-i}{3-4i}$. *Evaluate* $\tan[\tan^{-1}(\frac{4}{3}-\tan^{-1}(\frac{1}{7})]$.. *Hence find the principle argument of* $\frac{7-i}{3-4i}$ *in terms of* π.

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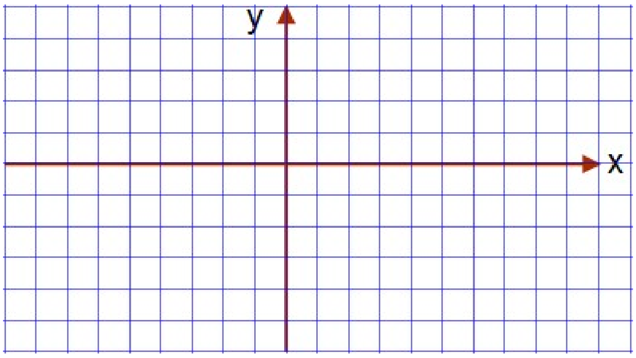
Exercise 1.7.7 *

1. *Solve the equation $x^2 + x + 1 = 0$ and $x^2 - \sqrt{3} + 1 = 0$. Plot on an Argand diagram the points A and B representing the solutions of the fist equation and C and D representing the solution of the second, choosing A and C to lie above the real axis.*



2. *Find the angle $\angle AOB,\ \angle COD,\ \angle VOA$ and $\angle ACB$.*

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