4 Unit Math Homework for Year 12

Student Name: ____________	**Grade**: ____________
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1 Topic 1 — Complex Numbers Part 1

1.1 The Arithmetic of Complex Numbers

1.1.1 The Structure of the Complex Number System

Definition: Consider the set $\mathbb{C}$ of numbers of the form $a + bi$
Where a and b are real, $i^2 = -1$
a is a real part, bi is an imaginary part.

Example 1.1.1 Find the sum and product of $2 + 3i$ and $3 + 5i$.

1. $(2 + 3i) + (3 + 5i)$.

Solution:
$$\begin{aligned}(2 + 3i) + (3 + 5i) &= 2 + 3 + 3i + 5i \\ &= 5 + 8i.\end{aligned}$$

2. $(2 + 3i) \times (3 + 5i)$

Solution:
$$\begin{aligned}(2 + 3i) \times (3 + 5i) &= 6 + 10i + 9i + 15i^2 \\ &= 6 + 19i - 15 \\ &= -9 + 19i.\end{aligned}$$

Exercise 1.1.1 Find the sum and product of $2 - 5i$ and $1 + 3i$

1. $(2 - 5i) + (1 + 3i)$

2. $(2 - 5i) \times (1 + 3i)$

1.1.2 Operations $+$, $-$, $\times$ and $\div$ on Complex Numbers

Addition: $(a+bi)+(c+di)=(a+c)+i(b+d)$
Subtraction: $(a+bi)-(c+di)=(a-c)+i(b-d)$
Multiplication: $(a+bi)\times(c+di)=ac+adi+cbi+bdi^2$
$$=(ac-bd)+i(ad+bc)$$
Division: $\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$.

Exercise 1.1.2 $z_1=2+3i$ **and** $z_2=1-2i$**. Evaluate the following:**

1. z_1+z_2

2. z_1-z_2

3. $z_1\times z_2$

4. z_1^2

1.1.3 Complex Conjugates and Reciprocals

Definition: The symbol $\overline{Z}$ denotes the complex conjugate of Z.

For example: $Z = 2 + 3i, \Rightarrow \overline{Z} = 2 - 3i$.

The product of complex conjugates is real.

For example: $(2 + 3i)(2 - 3i) = 4 - (-9) = 13$.

The modulus of a complex number Z is denoted $|Z| = |a + ib| = \sqrt{a^2 + b^2}$.

For example: $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

The reciprocal of the complex number Z is equal to its conjugate $\overline{Z}$, divided by the square of the modulus of the complex numbers Z.

The reciprocal of i is its own conjugate $-i$.

Example 1.1.2 Find the reciprocal of $3 - 4i$

Solution:

$$\begin{aligned}\frac{1}{3-4i} &= \frac{1 \times (3+4i)}{(3-4i) \times (3+4i)} \\ &= \frac{3+4i}{9+16} \\ &= \frac{3}{25} + \frac{4}{25}i\end{aligned}$$

Exercise 1.1.3 If $Z_1 = 4 + 2i$ **and** $Z_2 = 3 - 5i$**. Evaluate:**

1. $\frac{1}{Z_1}$

2. $\frac{1}{Z_2}$

3. $\frac{1}{Z_1^2}$

Exercise 1.1.4 $z_1 = 1 + 2i$ **and** $z_2 = 2 - 3i$ **Evaluate:**

1. $z_1 \times z_2$

2. $z_1\overline{z_1}$

3. $\frac{1}{z_1}$

4. $\frac{1}{z_2^2}$

5. $\frac{z_1}{z_2}$

Exercise 1.1.5 $z = -2 + 3i$**, Use** $\frac{1}{z} = \frac{\overline{z}}{z.\overline{z}}$ **to find** $\frac{1}{Z}$ **in the form** $a + ib,\ a, b \in \mathbb{R}$**.**

1.1.4 Real and Imaginary Parts of a Complex Number

Definition: If $z \in \mathbb{C}$, then z can be written in the form $z = x + iy, \;\; x, y \in \mathbb{R}$
The real part of z denoted $Re z$ and the imaginary part of z, denoted by $Im\, z$.

Example 1.1.3 $Z_1 = 1 + 2i$, $Z_2 = 2 - i$. **Find:**

1. $Re\,(z_1 + z_2)$

Solution: $z_1 + z_2 = 3 + i, \;\Rightarrow \therefore\; Re(z_1 + z_2) = 3$

2. $Im\,(z_1 z_2)$

Solution: $z_1 \times z_2 = (1 + 2i)(2 - i) = 2 - i + 2i - 2i^2 = 4 + i, \;\Rightarrow \therefore Im\,(z_1 z_2) = 1$

Exercise 1.1.6

1. *Find* $z \in \mathbb{C}$ *such that* $Re\, z = 2$ *and* z^2 *is imaginary.*

2. *Find* $z \in \mathbb{C}$ *such that* $Im\, z = 2$ *and* z^2 *is real.*

3. *Find* $z \in \mathbb{C}$ *such that* $Re\, z = 2Im\, z,$ *and* $z^2 - 4i$ *is real.*

1.1.5 Equality of Complex Numbers

Definition: If $a + bi = c + id,\ \ a, b, cd \in \mathbb{R}$ then

$$(a-c) + i(b-d) = 0 \ \Rightarrow \left\{\begin{array}{l} a - c = Re\,0 = 0 \\ b - d = Im\,0 = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} a = c \\ b = d \end{array}\right.$$

Example 1.1.4 Find the real a and b such that $x^3 - x + 2 = (x^2 + 1)Q(x) = ai + b$, for all x.

Solution: *Substituting* $x = i,\ \Rightarrow\ -i - i + 2 = 0 + ai + b$ *(since* $i^2 = -1)$
$\therefore 2 - 2i = b + ai,\ \ a\,b \in \mathbb{R}$
$\therefore b = 2$ *and* $a = -2$.

Exercise 1.1.7

1. *Given* $\frac{1+7i}{2-i} = a + bi,\ \ a,\ b \in \mathbb{R}$. *Evaluate* ab.

2. *If* $z_1 = 2 - i,\ \ z_2 = 1 - 3i$. *Find* $Im(\frac{i}{z_1} + \frac{z_2}{5})$

3. *Given* $z = i^2(1 + i)$. *Find Re(z).*

1.1.6 Solving Quadratic Equations with Real Coefficients

Definition: Given a quadratic equation $ax^2 + bx + c = 0$, its solution are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
To allow solution to exist for all values of a, b and c,
then we need to introduce the new number: $i = \sqrt{-1}; \Rightarrow$ ie. $i^2 = -1$
Every negative real number has two complex square roots.
For example: $-4 = 4i^2$ has square roots $2i$ and $-2i$.
Sum of two roots: $x_1 + x_2 = \frac{-b}{a}$; product of roots: $x_1 \times x_2 = \frac{c}{a}$.

Example 1.1.5 Solve $x^2 - 2x + 3 = 0$**.**

Solution: $\Delta = b^2 - 4ac = (-2)^2 - 4 \times 1 \times 3 = -8 = 8i^2$
$\therefore x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i.$

Exercise 1.1.8 Solve the following quadratic equations:

1. $x^2 + x + 1 = 0$

2. $2x^2 - 4x + 3 = 0$

Exercise 1.1.9 $3 + 2i$ **is one root of** $x^2 + bx + c = 0$ **where** $b, c \in \mathbb{R}$**. Find the value of** b **and** c**.**

1.1.7 Square Roots of Complex Numbers

Definition: In general, to find the square root of $a+ib,\ a,\ b \in \mathbb{R}, b \neq 0$

We solve $z^2 = a+ib$, where $z = x+iy,\ x,\ y \in \mathbb{R}$

Example 1.1.6 Given $z^2 = 3+4i$. Find z.

Solution: *Let* $z = x+iy,\ x,\ y \in \mathbb{R}$.

Then $(x+iy)^2 = 3+4i,\ \Rightarrow,\ (x^2-y^2)+(2xy)i = 3+4i$

Equating real and imaginary parts: $\begin{cases} x^2-y^2 = 3\ldots & (1) \\ 2xy = 4\ldots & (2) \end{cases}$

$\therefore x^4 - x^2y^2 = 3x^2$ *and* $x^2y^2 = 4$

Then $x^4 - 3x^2 - 4 = 0,\ \Rightarrow\ (x^2-4)(x^2+1) = 0,\ x \in \mathbb{R}$

$\therefore x = 2,\ y = 1,\ \Rightarrow z = 2+i$ *or* $x = -1,\ y = -1,\ \Rightarrow z = -2-i.$

Hence $3+4i$ *has two square roots* $2+i$ *and* $-2-i$.

Exercise 1.1.10 Find the square roots of the following complex numbers:

1. -25

2. i

3. $4-3i$

1.1.8 Solving Quadratic Equations with Complex Coefficients

Example 1.1.7 Solve $2x^2 + (1+i)x + (1-i) = 0$

Solution: *Find* Δ : $\Delta = b^2 - 4ac = (1-i)^2 - 8(1-i) = -8 + 6i$

Find square roots of Δ : *Let* $(a+bi)^2 = -8+6i,\ a,\ b \in \mathbb{R}$

Then $(a^2 - b^2) + (2ab)i = -8 + 6i$

Equating real and imaginary parts, $a^2 - b^2 = -8$ *and* $2ab = 6$

Now we have $a^2 - \frac{9}{a^2} = -8, \Rightarrow a^4 + 8a^2 - 9 = 0 \Rightarrow (a^2+9)(a^2-1) = 0$

a *real* $\Rightarrow a = 1, b = 3$ *or* $a = -1, b = -3.$

Hence Δ *has square roots* $1 + 3i$ *and* $-1 - 3i$

Use the quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(1+i) \pm (1+3i)}{4}$

$\therefore x = \frac{1}{2}i$ *or* $x = -\frac{1}{2} - i.$

Exercise 1.1.11 Solve the following quadratic equations:

1. $4x^2 - 4(1+2i)x - (3-4i) = 0.$

2. $ix^2 - 2(1+i)x + 10 = 0.$

Exercise 1.1.12 Given $z_1 = 2 - i,\ z_2 = 3 + 4i,\ z_3 = 2i$ **and** $z_4 = 2$**, find:**

1. $z_1 - z_2$

2. $z_2\overline{z_4}$

3. $Im(z_1 - 2z_2 + \overline{z_3})$.

4. $Re(\overline{z_1}z_3)$

5. $4z_1 - \frac{2\overline{z_2}}{z_3}$

Exercise 1.1.13 Prove the following results about complex conjugates:

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

2. $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$

Exercise 1.1.14

1. $x^2 + 6x + k = 0$ *has one root* α *where* $Im(\alpha) = 2$*. If* k *is real, find both roots of the equation and the value of* k*.*

2. $1 - 2i$ *is one root of* $x^2 - (3 + i)x + k = 0$*. find* k *and the other root of the equation.*

Exercise 1.1.15 Given $z_1 = 3 + 4i,\ z_2 = a + 6i$ and $z_3 = 3 + bi$, find if any values of a and b exist to allow for the following equations to hold:

1. $z_1 = z_2$

2. $z_1 = z_3$

3. $z_2 = z_3$

4. *Find a value of a and k, so that* $z_1 = kz_2$.

Exercise 1.1.16 If $z = a + ib$, find expressions for a and b in terms of z and $\overline{z}$.

1.2 Practical Exam Questions

Question 1 .... *(4 marks)*

Given $\frac{a+2i}{i} = b + i,\ (a, b \in \mathbb{R})$. Find the value of $a + b$.

Question 2 .... *(4 marks)*

If $z = 1 + i$, Evaluate $\frac{2}{z} + z^2$.

Question 3 .... *(11 marks)*

Evaluate $\frac{3+2i}{2-3i} - \frac{3-2i}{2+3i}$.

Question 4 .... *(4 marks)*

Given $z_1 = 4 + 29i$, and $z_2 = 6 + 9i$. Find the real part of $(z_1 - z_2)i$.

Question 5 .... *(2 marks)*

Find the square roots of the complex number $5 + 12i$.

Question 6 . . . . *(7 marks)*

If complex number z satisfied $z(1+i) = 1 - i$. Find its conjugate $\overline{z}$.

Question 7 . . . . *(6 marks)*

If cubic function $x^3 + ax^2 + bx - 2 = 0$ has a root equals to i, find the value of a and b.

Question 8 . . . . *(10 marks)*

Given $z = \ln(m^2 - 2m - 2) + (m^2 + 3m + 2)i$. Find the possible values of m which satisfy the following condition:

(a) z is an imaginary only;

(b) z is real;

(c) z is on the second quadrant.

1.3 Past HSC Exam Questions

Question 1 . . . . *(4 marks)*

Let $z = 5 - i$.

(a) Find z^2 in the form $x + iy$. [1]

(b) Find $z + 2\overline{z}$ in the form $x + iy$. [1]

(c) Find $\frac{i}{z}$ in the form $x + iy$. [2]

Question 2 . . . . *(4 marks)*

(a) Find all pairs of integers x and y such that $(x + iy)^2 = -3 - 4i$ [2]

(b) Hence or otherwise, solve the quadratic equation $z^2 - 3z + (3 + i) = 0$ [2]

Question 3 .... *(11 marks)*

(a) Find integers a and b such that $(x+1)^2$ is a factors of $x^5 + 2x^2 + ax + b$. [3]

(b) The equation $z^2 + (1+i)z + k = 0$ has root $1 - 2i$. Find the other root and the value k. [3]

(c) Find the values of real numbers a and b such that $(a+ib)^2 = 5 - 12i$. [2]

(d) Given that $z = 1 + i$ and $w = -3$, find, in the form $x = iy$:

i. wz^2, [1]

ii. $\frac{z}{z+w}$. [2]

Question 4 . . . . *(4 marks)*

Let $z = 3 - 2i$ and $u = -5 + 6i$

(a) Find $Im(uz)$ [1]

(b) Find $|u - z|$ [1]

(c) Find $\overline{-2iz}$ [1]

(d) Express $\frac{u}{z}$ in the form $a + ib$, where a and b are real numbers. [1]

Question 5 . . . . *(2 marks)*

Let $z = 2 + 3i$ and $w = 1 + i$. Find in the form $x + iy$, where x and y are real numbers.

(a) zw [1]

(b) $\frac{1}{w}$ [1]

Question 6 . . . . *(7 marks)*

(a) Let $z = 1 + 2i$ and $w = 1 + i$. Find in the form $x + iy,\ x,\ y\ \in \mathbb{R}$.

i. $z\overline{w}$ [1]

ii. $\frac{1}{z}$. [1]

(b) Let $z = 1 + 2i$ and $w = 3 - i$. Find, in the form $x + iy$,

i. zw [1]

ii. $\overline{\left(\frac{10}{z}\right)}$. [1]

(c) Let $z = 3 + i$ and $w = 2 - 5i$. Find, in the form $x + iy$,

i. z^2 [1]

ii. $\overline{z}w$ [1]

iii. $\frac{w}{z}$. [1]

Question 7 . . . . *(6 marks)*

(a) Let $z = 3 + i$ and $w = 1 - i$. Find, in the form $x + iy$,

i. $2z + iw$ [1]

ii. $\overline{z}w$ [1]

iii. $\frac{6}{w}$. [1]

(b) Given that $2 + i$ is a root of $P(z) = z^3 + rz^2 + sz + 20$, where $r,\ s \in \mathbb{R}$.

i. State why $2 - i$ is also a root of $P(z)$. [1]

ii. Factories P(z) over the real numbers. [2]