Year 11 Math Homework

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

Table of contents

This edition was printed on February 12, 2017 with worked solutions.
Camera ready copy was prepared with the LaTeX2e typesetting system.

9 Year 11 Topic 9 — The Derivative Part 4

9.1 The Derivative

9.2 Tangents and Normals

The normal is a line perpendicular to the tangent at the point where tangent touches the curve.

- Find the derivative and its value at the given point.
- Use this to find the gradient of the line.
- Use the point-gradient formula to find the equation.

Example 9.2.1 Find the equation of the tangent to $y = 3x^2 - 9x + 1$ **at the point where** $x = 2$

Solution: $y = 3x^2 - 9x + 1 \Rightarrow y' = 6x - 9 \Rightarrow$ *At* $x = 2 \Rightarrow y' = 6 \times (2) - 9 = 3$

$\therefore$ *the gradient of the tangent is 3.*

If $x = 2, \Rightarrow y = 3 \times (2)^2 - 9 \times 2 + 1 = -5 \Rightarrow$ *the point is* $P(2, -5)$

So $y - y_1 = m(x - x_1)$

$y - (-5) = 3(x - 2)$

$y + 5 = 3x - 6$

$\therefore$ *the required equation is* $3x - y - 11 = 0$

Example 9.2.2 Find the equation of the normal to $y = 2x^2 - 5x + 1$ **at** $x = 3$

Solution: $y = 2x^2 - 5x + 1 \Rightarrow y' = 4x - 5$ *At* $x = 3 \Rightarrow y' = 4 \times 3 - 5 = 7$

The gradient of the tangent is 7, $\therefore$ *the gradient of the normal is* $-\frac{1}{7}$.

If $x = 3 \Rightarrow, y = 2x^2 - 5x + 1 \Rightarrow y = 2 \times (3)^2 - 5 \times 3 + 1 = 4$

$y - y_1 = m(x - x_1)$

$y - 4 = -\frac{1}{7}(x - 3)$

therefore the equation of normal is $x + 7y - 25 = 0$

Example 9.2.3 Find the point on $y = 3x - x^2$ **where the tangent is horizontal.**

Solution: $y = 3x - x^2 \Rightarrow y' = 3 - 2x.$

Put $y' = 0$, *then* $3 - 2x = 0 \Rightarrow x = \frac{3}{2}$

Substituting $y = 3 \times (\frac{3}{2}) - (\frac{3}{2})^2 = \frac{9}{2} - \frac{9}{4} = \frac{9}{4} = 4\frac{1}{2}$

So the tangent is horizontal at $(\frac{3}{2}, 2\frac{1}{4})$

Exercise 9.2.1 Find the gradient of these lines:

1. *The tangent to* $y = 2x^3 - 5x + 7$ *at* $x = 2$.

2. *The normal to* $y = 3x - 3x^2 + 5$ *at* $x = -1$.

Exercise 9.2.2

1. *Find the equation of the tangent to the curve* $y = 3x^2 + x - 2$ *at the point where* $x = -1$.

2. *What is the equation of the normal at the point where* $x = 0$?

9.3 Parametric Differentiate

In many cases, a cure will be specified by two equations given x and y in term of some third variable t, called a parameter.

Parametric function: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Example 9.3.1 **If** $x = 2t$**, and** $y = t^2$**, find** $\frac{dy}{dx}$

Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t$

Exercise 9.3.1 **Use the parametric differentiation to find** $\frac{dy}{dx}$**, then evaluate** $\frac{dy}{dx}$ **when** $t = -1$:

1. $x = 5t$ *and* $y = 10t^2$

2. $x = ct$ *and* $y = \frac{c}{t}$

3. $x = at + b$ *and* $y = bt + a$

4. $x = 3t^2$ *and* $y = 2t^3$

9.4 Differentiating Inverse Functions

Suppose that y is a function of x, and that the inverse is also a function, so that x is a function of y. Then by **Chain Rule**: $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

Inverse Functions: $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ (provided neither is zero).

Example 9.4.1 Differentiate $y = x^{\frac{1}{2}}$; (a) directly, (b) by first forming the inverse function and then differentiating.

a. *Using the usual rule for differentiating powers of x,*

Solution: $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

b. *Solving for x,* $x = y^2$.

Solution: $\frac{dx}{dy} = 2y$, *and* $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2}x^{-\frac{1}{2}}$

Exercise 9.4.1

1. $y = x^{\frac{1}{5}}$

2. $y = \frac{1}{2-x}$

3. $y = \frac{1}{5x+3}$

9.5 Rates of Changes

The fractional notation for derivative as a ratio carries this interpretation of the derivative as a rate:

> Rates: Express one quantity as a function of the other quantity, then differentiate with respect to time using the chain rule.

Example 9.5.1 Suppose water is flowing into a large spherical balloon at a constant rate of $50\,cm^3/s$**.**

Notes that there are two quantities varying with time here, the volume and the radius:

- *The volume is increasing at a constant rate.*
- *the radius is increasing at a rate that decreases as the balloon expands.*

The ***chain rule*** *will allow the two rates of change to be related to each other.*

1. *At what rate is the radius* **r** *increasing when the radius is 7 cm?*

Solution: *The volume of a sphere is given by:* $V = \frac{4}{3}\pi r^3$
Differentiating with respect to t, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ *(chain rule)*
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
now we substituting the given rate of $\frac{dV}{dt} = 50$ *and the radius* $r = 7$*,*
$50 = 4\pi \times 49 \times \frac{dr}{dt}$
therefore the rate of increase of the radius: $\frac{dr}{dt} = \frac{25}{98\pi}$ *cm/s*

2. *At what rate is the radius increasing when the volume V is 4500*π cm^3*?*

Solution: *When* $V = 4500\pi$, $\frac{4}{3}\pi r^3 = 4500\pi \Rightarrow r^3 = 3375 \;\therefore r = 15$
so substituting again, $50 = 4\pi \times 225 \times \frac{dr}{dt} \Rightarrow \therefore \frac{dr}{dt} = \frac{1}{18\pi}$

3. *What should the flow rate be changed to so that when the radius is 7 cm, it is increasing at 1 cm/s?*

Solution: *Substituting* $r = 7$ *and* $\frac{dr}{dt} = 1$*, given the rate of change of volume:*
$$\frac{dV}{dt} = 4\pi \times 49 \times 1$$
$$= 196\ cm^3/s$$

Exercise 9.5.1 Given that $y = x^3 + x$, differentiate whit respect to time, using the chain rule.

1. *If $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when x = 2.*

2. *If $\frac{dy}{dt} = -6$, find $\frac{dx}{dt}$ when x =-3.*

Exercise 9.5.2 A circular oil stain of radius r and area A is spreading on water. Differentiate the area formula whit respect to time to show that $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Hence find:

1. *the rate of increase of area when r = 40 cm if the radius is increasing at $3\, cm/s$,*

2. *the rate of increase of the radius when r = 60 cm if the area is increasing at $10\, cm^2/s$.*

Exercise 9.5.3 A spherical bubble of radius r is shrinking so that its volume V is decreasing at a constant rate of $200\,cm^3/s$. Show that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

1. *At what rate is its radius decreasing when the radius is 5 cm?*

2. *What is the radius when the radius is decreasing at $2\pi\; cm/s$?*

3. *At what rate is the radius decreasing when the volume is $36\pi cm^3$*

9.6 Miscellaneous Exercises

Exercise 9.6.1 For each curve below: (i) find the equation of the tangent at the point P where x = a; (ii) hence find the equations of any tangents passing through the origin.

1. $y = x^2 - 10x + 9$

2. $y = x^2 + 15x + 36$

3. $y = 2x^2 - 7x + 6$

Exercise 9.6.2 By factorising top and bottom and cancelling, find these limits:

1. $\lim_{x\to 0} \frac{x^2-12x}{x^2+6x}$

2. $\lim_{x\to 3} \frac{x^2-3x}{x^2-9}$

3. $\lim_{x\to -2} \frac{x^2+5x+6}{x^2+6x+8}$

Exercise 9.6.3 Find the points on each curve where the tangent has the given angle of inclination:

1. $y = \frac{1}{3}x^3 - 7,\ 45^\circ.$

2. $y = x^2 + \frac{1}{3}x^3,\ 135^\circ$

Exercise 9.6.4 Differentiate using the most appropriate method. Factor each answer completely.

1. $\frac{x^2-x+1}{2\sqrt{x}}$

2. $\frac{2}{3}x^2(x^3-1)$

3. $(\sqrt{x}+\frac{1}{\sqrt{x}})^2$

4. $\frac{1}{x^3-4}$

5. $\frac{1}{\sqrt{3-2x}}$

Exercise 9.6.5

1. *Sum the arithmetic series* $1 + 3 + 5 + ...$ *to n terms.*

2. *Sum the arithmetic series* $2 + 4 + 6 + ...$ *to n terms.*

3. *Find the sum of the first 2000 terms of the series* $1 - 2 + 3 - 4 + 5 - 6 + (-1)^{n+1}n + ...$

4. *The line* $y = mx + b$ *is a tangent to the curve* $y = x^3 - 3x + 1$ *at the point (-2, -1). Find the values of* m *and* b*.*

9.7 Practical Exam Questions

Exercise 9.7.1

1. *A curve has parametric equations* $x = \frac{t}{2},\ y = 3t^2$. *Find the Cartesian equation for this curve.*

2. *Evaluate* $\lim_{x\to 0} \frac{3x}{sin2x}$.

3. *Evaluate* $\lim_{x\to 0} \frac{\sin(\frac{x}{5})}{2x}$.

4. *Differentiate* $5\sqrt{x} - \frac{10}{x62} - \frac{1}{2\sqrt{x}} + \pi$.