Year 11 Math Homework

Student Name: ____________	**Grade**: ____________
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9 Year 11 Topic 9 — The Derivative Part 1

9.1 The Derivative

9.1.1 Limits and Continuity

More formally, a function $f(x)$ is said to be continuous at $x = c$ if:

- $f(x)$ is defined at $x = c$,
- the limit of $f(x)$ as x approaches c exists,
- $f(x)$ is equal to this limit.
- Special limit: $\lim_{x\to\infty} \frac{1}{x} = 0$

Example 9.1.1

1. *Find* $\lim_{x\to 0} \frac{2x-1}{x+3}$

   Solution:

$$\lim_{x\to 0} \frac{2x-1}{x+3} = \frac{0-1}{0+3} = -\frac{1}{3}$$

2. *Find* $\lim_{x\to 3} \frac{x^2-9}{x-3}$

   Solution:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x\to 3}(x+3) = 6 \qquad (as\, x \neq 3)$$

3. *Find* $\lim_{x\to\infty} \frac{x^2-5x+1}{3x^2-1}$

   Solution:

$$\lim_{x\to\infty} \frac{x^2-5x+1}{3x^2-1} = \lim_{x\to\infty} \frac{1-\frac{5}{x}+\frac{1}{x^2}}{3-\frac{1}{x^2}} = \frac{1-0+0}{3-0} = \frac{1}{3}$$

Exercise 9.1.1 Evaluate the following limits:

1. $\lim_{x\to 3} \frac{x^2-5}{x+2}$

2. $\lim_{x\to 4} \frac{x-1}{x^2+x-2}$

9.1.2 Theorems on Limits

- Theorem 1: for the constant function f, where $f(x) = c$,

$$\lim_{x\to a} f(x) = c$$

- Theorem 2: If $\lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

- Theorem 3: If $\lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}(f(x) \times g(x)) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x) = L \times M$$

- Theorem 4: If $\lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}\ (if M \neq 0)$$

Exercise 9.1.2 Evaluate the following limits:

1. $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$

2. $\lim_{x\to 5} \frac{x-5}{2x^2-9x-5}$

3. $\lim_{x\to 0} \frac{(x+2)^2-4}{x}$

9.1.3 The Derivative — Geometric Definition

Let $f(x)$ be a function. The derivative or derived function of $f(x)$, written as $f'(x)$, is defined by:

Definition: $f'(x)$ is the gradient of the tangent to $y = f(x)$ at $P(x, f(x))$.

9.1.4 The Derivative as a Limit

The derivative as a limit is known as the *"first principles"*.

Definition:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 9.1.2 Differentiate $3x^2 - 5$ from first principles and find the gradient of curve $y = 3x^2 - 5$ where $x = -1$ and where $x = 3$.

Solution: *Let $P(x, y)$ be a point on the curve.*

Let $Q((x+\Delta x), (y+\Delta y))$ be a point near P on the curve as $y = 3x^2 - 5$.

Then $y + \Delta y = 3(x+\Delta x)^2 - 5$

$= 3(x^2 + 2x\Delta x + (\Delta x)^2) - 5$

$= 3x^2 + 6x\Delta x + 3\Delta x^2 - 5$

$\therefore \Rightarrow \quad \Delta y = 6x\Delta x + 3\Delta x^2$

$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = 6x + 3\Delta x$ *if* $\Delta x \neq 0$

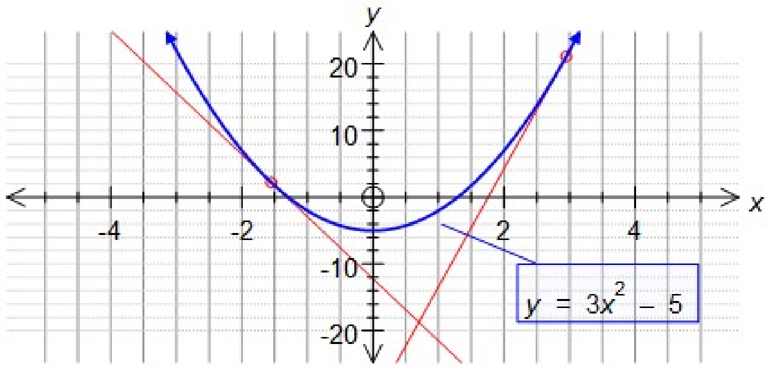
$\therefore \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 6x$

When $x = -1$, $\frac{dy}{dx} = -6$,

Therefore the gradient of the curve at $x = -1$ is -6

When $x = 3$, $\frac{dy}{dx} = 18$

Therefore the gradient of the curve at $x = 3$ is 18.



Example 9.1.3 Find the derivative of x^{-1} from first principles.

Solution: *Let* $y = x^{-1} = \frac{1}{x}$ $\Rightarrow$ *then* $y + \Delta y = \frac{1}{x+\Delta x}$

$$\Delta y = \frac{1}{x+\Delta x} - \frac{1}{x}$$

$$= \frac{x-(x+\Delta x)}{x(x+\Delta x)} = \frac{-\Delta x}{x(x+\Delta x)}$$

$$\frac{dy}{dx} = \frac{-1}{x(x+\Delta x)}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{-1}{x^2} \quad or \quad -x^{-2}$$

Example 9.1.4 Find the derivative of $x^{\frac{1}{2}}$ from first principles.

Solution: *Let* $y = x^{\frac{1}{2}}$ $\Rightarrow$ *then* $y + \Delta y = (x + \Delta x)^{\frac{1}{2}}$

$$\Delta y = (x + \Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}}{\Delta x}$$

$$= \frac{[(x+\Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}][(x+\Delta x)^{\frac{1}{2}} + x^{\frac{1}{2}}]}{\Delta x[(x+\Delta x)^{\frac{1}{2}} + x^{\frac{1}{2}}]}$$

$$= \frac{x+\Delta x - x}{\Delta x[(x+\Delta x)^{\frac{1}{2}} + x^{\frac{1}{2}}]}$$

$$= \frac{1}{(x+\Delta x)^{1/2} + x^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{(x+\Delta x)^{\frac{1}{2}} + x^{\frac{1}{2}}} = \frac{1}{2x^{\frac{1}{2}}} \quad or \quad \frac{1}{2}x^{\frac{1}{2}}$$

Exercise 9.1.3 Consider the functions $f(x) = x^2 - 6x$ and $f(x) = x^2 + 3x + 2$. Using the definition $f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, find their derivatives.

1. $f(x) = x^2 - 6x$

2. $f(x) = x^2 + 3x + 2$

Exercise 9.1.4 For each function below, simplify $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, then take $\lim_{\Delta x \to 0}$ to find the derivative.

1. $f(x) = x^2 - 4x$

2. $f(x) = 12 - 6x^2$

3. $f(x) = x^2 + 18$

Exercise 9.1.5 For each function below, simplify $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, then take $\lim_{\Delta x\to 0}$ to find the derivative.

1. $f(x) = 2x^2 + 3x + 6$

2. $f(x) = x^3$

3. $f(x) = \frac{x^2}{2} - 2x - 3$

9.1.5 Rules for Differentiation

Rule 1: The Derivative of Constant Function	$y = c$, $\frac{dy}{dx} = 0$ or $y' = 0$

Rule 2: The Derivative of a Linear Function	$y = mx$, $\frac{dy}{dx} = m$

Rule 3: The Derivative of Power of x	$y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

Rule 4: Linear Combination of Function	$y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$

Exercise 9.1.6 Find the derivatives of the following functions:

1. $f(x) = 5x - 1$

2. $f(x) = 12 + 7x$

3. $f(x) = 2ax^3 + 12x - 8$

4. $f(x) = 3x^{\frac{2}{3}}$

5. $f(x) = 4x^{-5}$

6. $f(x) = 2x^{-\frac{1}{2}}$

Exercise 9.1.7 Write each function in the form of $f(x) = mx + b$, and hence write down $f'(x)$.

1. $f(x) = 3x + 5$

2. $f(x) = 5 - 3x$

3. $f(x) = 9 - 6x^2$

4. $f(x) = \frac{2}{3}(x + 6)$

5. $f(x) = \frac{3-4x}{5}$

6. $f(x) = \frac{3}{2}(6 - \frac{4}{3}x)$

7. $f(x) = \frac{1}{5x} - 3$

Exercise 9.1.8 Use the rule for differentiating ax^n to differentiate the following functions (where a, b, and c are constants).

1. $f(x) = 8x^3$

2. $f(x) = x^3 + x^2 + x - 4$

3. $f(x) = x^{4a+1}$

4. $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 5$

5. $f(x) = ax^3 + bx^2 - c$

6. $f(x) = ax^{2a}$

7. $f(x) = (ax + 3b)^2$

Exercise 9.1.9 Differentiate the following functions:

1. $f(x) = \frac{2}{3x^3}$

2. $f(x) = \sqrt{4x}$

3. $f(x) = 6x^3 - 3x^2 + 2$

4. $f(x) = \frac{6}{x^2} + \frac{2}{x^3}$

5. $f(x) = \frac{1}{6}x^4 - \frac{1}{2}x^2$

6. $f(x) = \frac{2}{3}(3x^3 - x^2 + 4)$

7. $f(x) = \frac{2x}{3}(1 + \frac{1}{3}x^2)$

9.1.6 Practical Exam Questions

Exercise 9.1.10 Differentiate the following functions:

1. $f(x) = \sqrt{x\sqrt{x\sqrt{x}}} - \frac{1}{\sqrt{x\sqrt{x\sqrt{x}}}}$

2. $f(x) = (x+1)(x+2)(x+3)$

3. $f(x) = \frac{1}{1-\sqrt{x}} + \frac{1}{1+\sqrt{x}}$

4. $f(x) = \frac{4x^3+3x^2+2x+1}{x}$