Year 11 Math Homework

Student Name: ____________	**Grade**: ____________
Date: ____________	**Score**: ____________

Table of contents

This edition was printed on October 25, 2011.

Camera ready copy was prepared with the LaTeX2e typesetting system.

17 Topic 17 — Polynomials (Part 1)

A polynomial in x is an algebraic expression involving the sum of many terms. For example: Binomial involving only 2 terms, Trinomial involving 3 terms.

Definition: A polynomial P(x) of degree n is an algebraic expression of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $n, n-1, n-2 \ldots$ are positive integers and $a_n,\ a_{n-1},\ a_{n-2}$ are real numbers.

17.1 Addition, Subtraction and Multiplication of Polynomials

Example 17.1.1

1. *Find the sum of the polynomial* $P(x) = 3x^2 + 2x - 5$ *and* $Q(x) = 2x^2 - 4x + 9$.

Solution: $Sum = P(x) + Q(x)$
$= (3x^2 + 2x - 5) + (2x^2 - 4x + 9)$
$= 5x^2 - 2x + 4$

2. *Find the difference of the polynomials* $f(x) = 4x^3 - 2x^2 + 5$ *and* $Q(x) = x^2 + 5x - 2$.

Solution: $Difference = P(x) - Q(x)$
$= (4x^3 - 2x^2 + 5) - (x^2 + 5x - 2)$
$= 4x^3 - 3x^2 - 5x + 7$

3. *Find the product of the polynomials* $5x^2 + 3x - 7$ *and* $2x^2 - 2x + 2$.

Solution: $Product = (5x^2 + 3x - 7) \times (2x^2 - 2x + 2)$
$= 5x^2(2x^2 - 2x + 2) + 3x(2x^2 - 2x + 2) - 7(2x^2 - 2x + 2)$
$= 10x^4 - 10x^3 + 10x^2 + 6x^3 - 6x^2 + 6x - 14x^2 + 14x - 14$
$= 10x^4 - 4x^3 - 20x^2 + 20x - 14$

17.2 Sketching Graphs of Factorised Polynomials

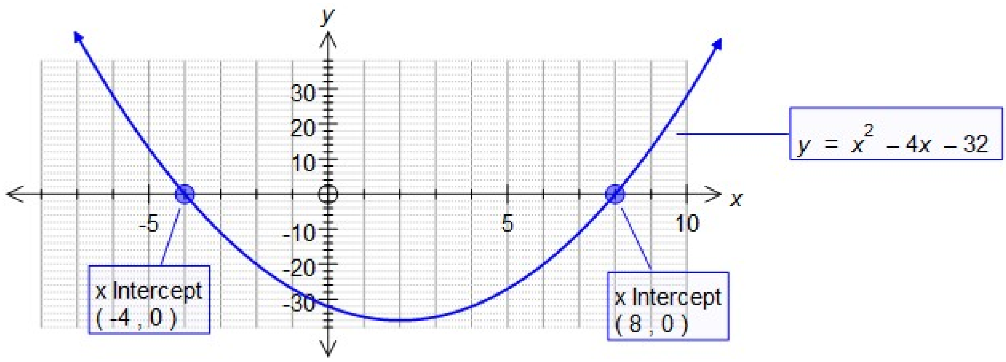
To sketch the graphs of factorised polynomials:

- Determine the zeros by equating each bracket equal to zero.
- Determine the degree of the polynomial, and then use the basic shape to sketch the graph.

Example 17.2.1

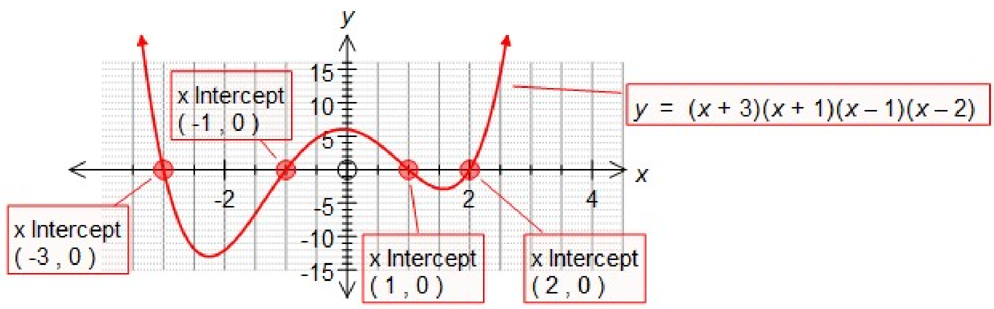
1. *Find the zeros of the polynomial* $f(x) = x^2 - 4x - 32$

Solution: *Factoring the quadratic equation* $x^2 - 4x - 16 = (x-8)(x+4) = 0$
$\therefore x = -4$ *and* $x = 8$ *are the zeros. the graph is a simple parabola.*



2. *Find the zeros of the polynomial* $f(x) = (x+3)(x+1)(x-1)(x-2)$ *and draw a sketch of the graph.*

Solution: $(x+3)(x+1)(x-1)(x-2) = 0$ $\therefore$ *the zeros are* $x = -3,\ -1,\ 1$ *and* 2.



17.3 Division of Polynomials

In General: It is useful to express the division process as $P(x) = A(x)Q(x) + R(x)$
where $P(x)$ is the original polynomial,
$A(x)$ is the divisor,
$Q(x)$ is the quotient
$R(x)$ is the remainder.

Example 17.3.1 Complete the division and express the polynomial $(3x^3 - 4x^2 + 2x + 4) \div (x - 3)$ **in the form** $P(x) = A(x)Q(x) + R(x)$**.**

Solution:

$$\begin{array}{r|l} & 3x^2 \quad + 5x + 17 \\ \hline x - 3) & 3x^3 - 4x^2 \quad + 2x \quad + 4 \\ & -\, 3x^3 + 9x^2 \\ \hline & \qquad 5x^2 \quad + 2x \\ & \qquad -\, 5x^2 + 15x \\ \hline & \qquad\qquad 17x \quad + 4 \\ & \qquad\qquad -\, 17x + 51 \\ \hline & \qquad\qquad\qquad 55 \end{array}$$

$\therefore (3x^3 - 4x^2 + 2x + 4) \div (x - 3) = 3x^2 + 5x + +17$ *plus remainder* $\frac{55}{x-3}$
If we multiply both sides by $(x - 3)$*, we can write the solution as:*
$3x^3 - 4x^2 + 2x + 4 = (x - 3)(3x^2 + 5x + +17) + 55.$

Exercise 17.3.1 Complete the division and express $(3x^2 + 15x + 8) \div (x + 3)$ **in the form** $P(x) = A(x)Q(x) + R(x)$**.**

Exercise 17.3.2 Complete the division and express them in the form $P(x) = A(x)Q(x) + R(x)$**.**

1. $(2x^3 + 3x^2 + 5x - 4) \div (2x + 1)$.

2. $(6x^2 - 9x + 10) \div (x - 1)$.

3. $(x^3 - 2x^2 + 4x + 6) \div (x - 3)$.

Example 17.3.2 Determine the quotient and remainder for $(x^4 + 4x^3 + 3x + 5) \div (x^2 + 2x + 1)$

Solution:

$$
\begin{array}{rl}
 & \; x^2 \quad + 2x \quad - 5 \\
\cline{2-2}
x^2 + 2x + 1 \big) & x^4 + 4x^3 \qquad\qquad + 3x \quad + 5 \\
 & -x^4 - 2x^3 \quad - x^2 \\
\cline{2-2}
 & \qquad 2x^3 \quad - x^2 \quad + 3x \\
 & \quad -2x^3 - 4x^2 \quad - 2x \\
\cline{2-2}
 & \qquad\qquad -5x^2 \quad + x \quad + 5 \\
 & \qquad\qquad\quad 5x^2 + 10x \quad + 5 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 11x + 10
\end{array}
$$

$\therefore x^4 + 4x^3 + 3x + 5 = (x^2 + 2x + 1) \times (x^2 + 2x - 5) + (11x + 10)$

Exercise 17.3.3 Determine the quotient and remainder for these divisions:

1. $(5x^4 - 3x^3 + 2x - 4) \div (x^2 - 2x)$.

2. $(x^4 - 3x^2 + 5x - 5) \div (x^2 + 5)$.

17.4 The Remainder Theorem

Definition: If a polynomial $P(x)$ is divided by $(x-a)$
then the remainder can be found by evaluating $P(a)$

Example 17.4.1 Find the remainder when $f(x) = x^4 - 2x^3 + 5x + 2$ **is divided by** $x - 2$**.**

Solution: *Use the Remainder Theorem and evaluate* $f(2)$*:*
$f(2) = (2)^4 - 2 \times (2)^3 + 5 \times (2) + 2 = 12, \Rightarrow \therefore$ *Remainder is 12.*

Example 17.4.2 Find the remainder when $f(x) = 3x^2 - 4x^2 - 35x + 12$ **is divided by** $x + 3$.

Solution: *Use the Remainder Theorem and evaluate* $f(-3)$*:*
$f(-3) = 3 \times (-3)^3 - 4 \times (-3)^2 - 35(-3) + 12 = 0$
$\therefore$ *Remainder is* 0*.*
Since $(x+3)$ *divided into f(x) with no remainder, it must be an exact factor of f(x).*

Exercise 17.4.1 Use the remainder theorem to find the remainder in each case:

1. $(3x^2 + 4x + 5) \div (x + 3)$

2. $(2x^4 - 5x^2 + 8x - 6) \div (x - 2)$

3. $(x^4 + 2x^3 + 6x - 4) \div (x + 4)$

17.5 The Factor Theorem

Definition: Consider any polynomial P(x) If P(a) = 0, then (x-a) is a factor of P(x)
Steps in factoring:

1. Find the first factor $(x - a)$ using a trial and error approach.
   Try $P(1),\ P(-1), P(2),\ P(-2)\, etc.$ Until the remainder is zero.
2. The other factor $Q(x)$ can then be found by dividing the polynomial $P(x)$ by the fist factor $(x - a)$.
3. If this other factor is a quadratic, then it can be simply factorised by inspection in the usual way.
4. If this factor $Q(x)$ is a polynomial of degree higher than 2, then steps (1) and (2) are repeated for this other factor.

Example 17.5.1 Find the factors of $x^3 + 3x^2 + 7x + 10$

.

Solution: *Try* $P(1) = (1)^3 + 3(1)^2 + 7(1) + 10 = 21 \neq 0 \;\Rightarrow\; (x - 1)$ *is not a factor.*
Try $P(-1) = (-1)^3 + 3(-1)^2 + 7(-1) + 10 = 25 \neq 0 \;\Rightarrow\; (x + 1)$ *is not a factor.*
Try $P(2) = (2)^3 + 3(2)^2 + 7(2) + 10 = 44 \neq 0 \;\Rightarrow\; (x - 2)$ *is not a factor.*
Try $P(-2) = (-2)^3 + 3(-2)^2 + 7(-2) + 10 = 0 \;\Rightarrow\; (x + 2)$ *is a factor.*

$$
\begin{array}{r|l}
 & x^2 \quad + x \quad + 5 \\
x + 2 & x^3 + 3x^2 + 7x + 10 \\
 & -x^3 - 2x^2 \\
 & \qquad x^2 + 7x \\
 & \qquad -x^2 - 2x \\
 & \qquad\qquad 5x + 10 \\
 & \qquad\qquad -5x - 10 \\
 & \qquad\qquad\qquad 0
\end{array}
$$

$\therefore x^3 + 3x^2 + 7x + 10 = (x^2 + x + 5)(2 + x)$

Exercise 17.5.1 Find the factors of the following polynomials:

1. $x^3 + 3x^2 - 4x - 12$

2. $x^3 + 2x^2 - x - 2$

Exercise 17.5.2 Show that the following division have a remainder of zero, and hence write the dividend as a product of its factors:

1. $(x^4 + 3x^3 - x^2 - 3x) \div (x^2 - 1)$

2. $(x^5 + 4x^4 - x^3 - 16x^2 - 12x) \div (x^2 + 4x + 3)$

3. $(3x^5 + 12x^4 + 2x^3 - 16x^2 - 8x) \div (x^2 + 4x + 2)$

Example 17.5.2 When $x^4 + 2x^3 + 3x + k$ is divided by $(x + 3)$ the remainder is 23. Find k.

Solution:

$$
\begin{array}{r|llll}
 & x^3 & -x^2 + 3x & & -6 \\
\hline
x+3) & x^4 + 2x^3 & & +3x & +k \\
 & -x^4 - 3x^3 & & & \\
\hline
 & -x^3 & & & \\
 & x^3 + 3x^2 & & & \\
\hline
 & & 3x^2 + 3x & & \\
 & & -3x^2 - 9x & & \\
\hline
 & & & -6x & +k \\
 & & & 6x & +18 \\
\hline
 & & & & (18 + 1k)
\end{array}
$$

As $18 + k = 23, \ \Rightarrow \ k = 5.$

Or from the factor theorem, if $(x + 3)$, *then* $f(-3) = 0$

If it is divided by a factor $(x + 3)$ *with a remainder 23, then:*

$f(-3) = x^4 + 2x^3 + 3x + k = 23$

$(-3)^4 + 2(-3)^3 + 3(-3) + k = 23$

$81 - 54 - 9 + k = 23$

$\therefore k = 5$

Exercise 17.5.3

1. *For what value of k is* $x + 1$ *a factor of* $x^3 - kx + 3$*?*

2. *When* $kx^4 + 3x^2 - kx + 1$ *is divided by* $(x - 2)$ *the remainder is 41. Find k.*